THE CAMBRIDGE AND DUBLIN MATHEMATICAL JOURNAL.

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VOL. III.

(BEING VOL. VII. OF THE CAMBRIDGE MATHEMATICAL JOURNAL.)

Δυῶν ὀνομάτων μορφὴ μία.

CAMBRIDGE:
MACMILLAN, BARCLAY, AND MACMILLAN;
GEORGE BELL, LONDON;
HODGES AND SMITH, DUBLIN.

1848

CAMBRIDGE:
Metcalfe and Palmer, Printers, Trinity Street.

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Notes and other passages enclosed in brackets [ ], are insertions made by the Editor.

The date of actual publication of any article, or portion of an article, may be found by referring to the first page of the sheet in which it is contained.—ED.

ERRATA.

Page 46, last sentence of foot note, dele " and of those points where the motion of the generating curve is in its osculating plane," and insert at the end of the note—if the case considered be that of a surface of revolution generated by the motion of a meridian curve.

Page 84, in an article entitled "Theorems on the Solution of a certain Partial Differential Equation," wherever $4\pi\rho$ occurs, it is with the wrong sign. Readers are requested to make the necessary correction.

Page 132, second line from foot, *for* ελεκτρον *read* ηλεκτρον.

Page 140, second line from foot, *for* série *read* série.

Page 208, lines 8, 9, 10, *for* arc $= c \cos \lambda . \theta$, &c. *read* arc $= c\theta$, radius $= c \cos \lambda$. Hence deviation $= \dfrac{c\theta^2}{2 \cos \lambda}$. In the great circle arc $= c\theta$, radius $= c$. Hence deviation $= \dfrac{c\theta^2}{2}$.

— line 17, *read* $\dfrac{c\theta^2}{2} \sqrt{\left(1 + \dfrac{1}{\cos^2 \lambda} - 2\right)} = \dfrac{c\theta^2}{2} \dfrac{\sin \lambda}{\cos \lambda} = \dfrac{c\omega^2 t^2}{2} . \sin \lambda \cos \lambda$.

This expression should have been used in finding the numerical results.

THE

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MATHEMATICAL JOURNAL.

ON A PRINCIPLE IN THE THEORY OF SURFACES OF THE SECOND ORDER, AND ITS APPLICATION TO M. JACOBI'S METHOD OF GENERATING THE ELLIPSOID.

By R. Townsend.

It is a long and familiarly known theorem in the Geometry of Plane Curves, that a curve of the second order may be generated by the motion of a point in a plane whose distance from a fixed point (called the focus) is to its distance from a fixed right line (called the directrix) measured in a direction parallel to a fixed directing right line in a constant ratio (called the modulus or modular ratio); and that, by altering the relative positions of the focus, directrix, and directing right line, as well as the magnitude of the modular ratio, every variety of curve of that order, except the circle and two parallel right lines, may be obtained by this, which has been called, for the sake of distinction, *the modular method* of generation.

An exactly analogous theorem in the Geometry of Surfaces has been discovered by Professor MacCullagh. He has shewn* that a surface of the second order may be generated by the motion of a point in space, whose distance from a fixed point (which by analogy he has called a focus) is to its distance from a fixed right line (which he has called a directrix), measured in a direction parallel to a fixed directing plane, in a constant ratio (which he has also called the modulus or modular ratio). And that, by altering the relative positions of the focus, directrix, and directing plane, as well as the magnitude of the modular ratio, every variety of sur-

* For a full development of Professor MacCullagh's theory, see *Proceedings of the Royal Irish Academy*, Part VIII. An abstract of his method will be found in the first volume of the same series, page 89.

face of that order, except the sphere, the *prolate** surfaces of revolution, and two parallel planes, may be obtained by this, which has been called by its illustrious author *the modular method* of generation.

It is evident, *a priori*, that, as in the former case, whatever be the nature of the generated curve, it must be symmetrical on both sides of the right line drawn through the focus perpendicular to the directrix; so in the present case, whatever be the nature of the generated surface, it must be symmetrical on both sides of the plane drawn through the focus perpendicular to the directrix, for drawing through the line in which that perpendicular plane intersects the directing plane a third plane, making with the former an angle equal but at the opposite side to that made with the same by the directing plane, it is obvious that the distances of the generating point from the directrix, in place of being measured parallel to the old, might be measured parallel to this new directing plane, and that their magnitudes would remain unaltered. Again, it is also obvious, *a priori* (as Professor MacCullagh has observed), that whatever be the nature of the generated surface, its sections by planes parallel to either of these two directing planes, must always be either all circles or all right lines, according as the modular ratio differs from or is equal to unity: for, taking the point in which any one individual plane of either of these two parallel systems intersects the directrix, the surface locus of all the points in space, whose distance from that point is to its distance from the focus in any constant ratio, will be a sphere or a plane according as the ratio is of inequality or equal to unity, and will therefore intersect the parallel plane in a circle or in a right line, as the case may be.

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* The term *oblate*, though very appropriate as applied to the *modular* ellipsoid of revolution, is far from being equally applicable either to the circular hyperboloid of one sheet, the modular hyperboloid of revolution, or to the parabolic cylinder, the modular paraboloid of revolution, which surface may always be considered as generated by the revolution of a parabola round its infinitely distant secondary axis. It would therefore be perhaps convenient to distinguish as a class these three species of surface of revolution, viz. those generated by the revolution of the three species of conic round their secondary axes by the common name of *modular* surfaces of revolution, inasmuch as they may be all obtained by the modular method of generation by simply taking the directrix perpendicular to the directing plane; while, on the contrary, the remaining three species, those generated by the revolution of the three species of conic round their primary axes, cannot by any variation of the circumstances be obtained by that method at all. A common name on grounds equally strong may, as we shall presently see, be also suggested for distinguishing as a class the three latter species, to *all* of which, for the same obvious reason, the term *prolate* is far from being applicable.

Hence, as in the modular generation of curves of the second order, the fixed point or focus is always on a principal axis of the curve, and the corresponding directrix is always perpendicular to the same principal axis; so in the modular generation of surfaces of the second order the fixed point or focus is always in a principal plane of the surface, and the corresponding directrix is always perpendicular to the same principal plane. And again, the sections of the surface by planes parallel to the two directing planes being either all circles or all right lines, the intersecting lines of every pair of these two different systems of parallel planes are always parallel all to one of the principal axes of the surface. The two particular planes passing through that axis itself have been called by Professor MacCullagh the *directive planes* of the surface, that appellation having been preferred to the name *cyclic*, as including the case of the hyperbolic paraboloid, as serving to keep in view the important position which these two planes occupy in the modular generation of the surface: the axis in which they intersect has been called by the same author the *directive axis* of the surface, and that axis coincides always, in the ellipsoid with the *mean* axis of the surface, in the hyperboloid of one sheet with the *primary* axis, and in the hyperboloid of two sheets with the *secondary* axis.

Moreover, Professor MacCullagh has shewn that, as in the modular generation of curves of the second order there exist two different foci both on the same principal axis, and two different directrices corresponding to these foci, both perpendicular to the same axis by which, with the same modular ratio and the same directing right line, the same curve may be generated; so also in the modular generation of surfaces of the second order, the focus need not be a unique point nor the directrix a unique right line, but that there always exists an infinite number of foci all in the same principal plane, and also an infinite number of corresponding directrices all perpendicular to the same plane, by which with the same modular ratio and the same directive planes the same surface may be generated. He has also shewn that the plane curve locus of the whole system of foci is always of the second order, and is no other than one of the well-known *focal conics** of the generated surface, so that the name as

* If, with Professor MacCullagh, we take as our *definition* of one of the focal conics in every surface of the second order, the locus of the system of foci in the modular method of generation, and as our definition of another

applied to that particular focal conic is peculiarly applicable and happy; and that the *cylinder* locus of the whole system of corresponding directrices is also always of the second order, and is always connected with the conic locus of the system of foci by a very simple, elegant, and remarkable relation. The curve in relation to the generated surface he has called the *modular focal conic,* and the cylinder in relation to the same he has called the *modular dirigent cylinder.*

In curves of the second order each focus and its corresponding directrix are always connected by the well-known relation of being reciprocally pole and polar to each other with respect to the curve. Similarly, in surfaces of the second order the modular focal conic and its corresponding dirigent cylinder have been shewn by Professor MacCullagh to be always connected by the perfectly analogous relation of being polars reciprocal to each other with respect to the surface, each individual directrix being always the polar of the line which at the corresponding focus is a tangent to the focal conic, and the plane containing both focus and directrix being always perpendicular to that focal tangent; and therefore also the base of the modular dirigent cylinder, or, as it has been called with respect to the surface, the *modular dirigent conic,* being always the polar reciprocal of the corresponding modular focal conic with respect to the principal section of the surface in their common principal plane.*

In the hyperboloid of one sheet and in the hyperbolic paraboloid the two real focal conics are always both modular, and therefore in connection with these two surfaces the two real dirigent cylinders and the two real dirigent conics are also both always modular; but in the ellipsoid, the elliptic paraboloid, and the hyperboloid of two sheets, but one of the two real focal conics, and therefore but one of the real dirigent cylinders and of the real dirigent conics, is ever

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the locus of the system of foci in another method of generation, due to Mr. Salmon, Fellow of Trinity College, Dublin, which we shall presently notice, it can be shewn from the position, equation, and properties of the resulting curves, that they are identical with those which we obtain by taking the boundaries of the infinitely flat surfaces of the second order confocal with the original surface. This has been shewn by both these authors on the principles of their repective methods, and it will be also established in the course of the present paper.

* To avoid delay, let these different focal properties of the modular method, and also those that correspond to them in Mr. Salmon's method of generation, which shall be immediately noticed, be for the present assumed by the reader; both systems of properties, as we proceed, will be proved by a single geometrical principle common to all focal conics, a principle which it is the chief object of the present paper to establish and illustrate.

modular, while the other, with its corresponding dirigent cylinder and conic, is altogether unconnected with the modular method of generation, and neither itself possesses any of the properties involved in and resulting from that consideration, nor can any of its points be employed as a focus in that method of generation. A modular focal conic in any surface of the second order may be always distinguished by the property that it never meets the surface, except in the particular case at the vertex of a cone; for, if the distance of any point on the surface from a point on the modular focal conic were nothing, then, making that point focus, the distance of the same from the corresponding directrix should also vanish, and therefore the focus should be on the directrix, in which case the surface would obviously be a cone, either in its general or in some modified form. On the contrary, the non-modular focal conic when real, wherever it exists, always intersects the surface, and therefore, since in its passage through that conic always pierces the umbilici, it has, as well for that reason as for the sake of distinction, been called the *umbilicar focal conic*.

But here it is to be carefully observed that in the enumeration just given the *real* focal conics alone have been taken into account: this was all very well so long as we were speaking only of the practical generation of the different curves and surfaces, for which purpose imaginary foci would in either case be of course unavailing; but theoretically speaking the imaginary foci and focal conics are actually included in the modular generation of curves and surfaces of the second order, and must moreover be also taken into account if we wish to extend to them the real focal properties involved in and resulting from the consideration of that method. Hence, then, taking into account the imaginary as well as the real focal conics, we may state the law of their distribution as modular or non-modular in a more general and satisfactory form, holding equally for all surfaces of the second order, and not as above, apparently expressing for one a property not possessed by another: in this point of view two of the three focal conics in every surface of the second order are always modular, and the third is always umbilicar, the two modular focals, whether both real or one imaginary, lying always in the two principal planes of the surface which intersect in the directive axis, and the single umbilicar focal, real or imaginary, lying always in the principal plane perpendicular to that axis. Hence, in every surface of the second order one

at least of the modular focal conics is always real, and therefore available for the purpose of generation by the modular method; but theoretically speaking they all admit of a double modular generation corresponding to the two modular focals, and with generally two different moduli, one for each focal. The modulus corresponding to a real modular focal is always real when the surface itself is real, but the modulus corresponding to an imaginary focal is not necessarily imaginary; for (as we shall presently see) an imaginary modular ratio could out of all the surfaces of the second order produce only the ellipsoid, which in passing through its limit the elliptic paraboloid into the hyperboloid of two sheets would cause that ratio to pass through infinity and become real, the corresponding focal conic remaining all the while imaginary. And in all the surfaces indifferently, if we denote by m and n the two different moduli, and by ϕ and $90^\circ - \phi$ the angles which the directive planes of the surface make with the principal planes of the corresponding focal conics respectively, Professor MacCullagh has shewn that those quantities are in all cases connected by the equation

$$\frac{\cos^2\phi}{m^2} + \frac{\sin^2\phi}{n^2} = 1;$$

a result easily verified, if, remembering that the different quantities are all constant for the same surface, we choose convenient particular points for the purpose of determining and comparing their actual values.

As in curves of the second order generated by the modular method the species of the resulting curve may be always determined *à priori*, independently of the arbitrary positions of the focus and directrix, when we have the modular ratio and the common inclination of the directing right lines to the corresponding axis of the curve, so in surfaces of the second order Professor MacCullagh has shewn that the species of the resulting surface may be always to a certain extent determined *à priori*, independently of the arbitrary positions of the focus directrix and directive axis, when we have the generating modular ratio and the common inclination of the directing planes to the corresponding principal plane of the surface. To see this, denoting by m and ϕ the modular ratio and corresponding inclination, let the principle of the modular generation be thrown by the reader into algebraical language, the focus and directrix being considered as both real, but as placed quite arbitrarily and not in any particular positions relatively

either to each other or to the intersection of the directing planes, which would of course give only particular surfaces, and in the resulting equation let m^2 be followed in its transitions through all successive values, from $+\infty$ to 0, and it will be immediately evident to mere inspection that, from $m=\infty$ to $m=1$, the resulting surface will be always an hyperboloid of *one* sheet, and the focus of generation will be always on its focal *ellipse:* for $m=1$, that the surface and focal will always pass each through one of its particular and limiting forms, the former being then always an *hyperbolic* paraboloid, and the latter consequently a *parabola:* for m, after having passed through 1, that the surface having merely passed through a particular form will still for some time be an hyperboloid of *one* sheet, but placed in a different position with respect to the focus and directrix, the real and imaginary non-directive axes having during the transition mutually interchanged their respective positions; and that the focal having passed from an ellipse through a parabola becomes and remains for some time an *hyperbola:* for some particular value of m intermediate to 1 and $\cos\phi$, but depending as to its actual magnitude on the relative positions of the focus, directrix, and directive axis, that the surface and focal will both simultaneously reach their respective *asymptotic* limits, the former becoming a *cone*, and the latter the two focal *right lines*, that is, the focal hyperbola of that particular surface: from that particular value of m down to $m=\cos\phi$, that the surface changing its species becomes and remains an hyperboloid of *two* sheets, having passed from an hyperboloid of one sheet through the cone, the common asymptotic limit to the two species of hyperboloid; and that the focal undergoing an analogous change passes from an hyperbola, through two right lines, into *another hyperbola* conjugate in position to the former and placed in the supplemental angle between the two right lines, the common asymptotic limit to an hyperbola and its conjugate: for $m=\cos\phi$, that the surface and focal again reach simultaneously their respective limiting and transition states, the former degenerating into an *elliptic* paraboloid, and the latter into its internal *parabola:* and finally, from $m=\cos\phi$ to $m=0$, that the surface and focal, having passed from an hyperboloid and an hyperbola respectively through the transition states of a paraboloid and a parabola, become and remain an *ellipsoid* and an *ellipse* respectively: after this, m^2 having reached the particular value of zero, may certainly pass on through that value; but it would be useless for us

to follow it farther, inasmuch as an imaginary modular ratio with a real focus and directrix could of course produce only an imaginary surface.

Thus, from the above enumeration, we see that with a real focus directrix and modular ratio we can get, by the modular method of generation, all the five principal surfaces of the second order in their general state, just as in the analogous method for conics we can get all the three principal curves of the second order in their general state; and also from the same we perceive very distinctly the nature and origin of the double modular generation in the hyperboloid of one sheet and in the hyperbolic paraboloid. Moreover, by observing the nature of the resulting surface during the changes of m, we see with a clearness which could not be surpassed, how by the variation of a single arbitrary quantity the six different species of surface of the second order, thus resulting all from a single and simple geometrical conception, gradually approach, enter upon, and pass through the respective states of transition from one to the other, and flow as it were into each other from species to species;—how that, commencing with the hyperboloid of one sheet, that surface, by the expansion of one of its principal hyperbolas in receding towards infinity and by the elongation of the other and of its principal ellipse while both remain finitely distant, approaches, reaches, and passes through its particular form, the hyperbolic paraboloid, while the expanding hyperbola, approaching, reaching, and passing through infinity both in magnitude and position, contracts in the supplemental angle between its asymptotes into the conjugate hyperbola, and while the other two principal sections, approaching, reaching, and passing through two parabolas, simultaneously interchange their respective species;—how after this the surface, having passed through that particular form, approaches, by the simultaneous contraction of its three principal sections, enters upon, and passes through its asymptotic limit the cone, at which moment its principal ellipse dwindles into a point and its two hyperbolas both contract into pairs of right lines; after which, by the ellipse having passed through nothing becoming imaginary, and by the two hyperbolas having passed through pairs of right lines opening out into the conjugate hyperbolas, that limiting surface expands into an hyperboloid of two sheets;—how that then the surface in its new form, by the expansion and recession towards infinity of its principal ellipse and by the elongation one from the other of its two principal hyperbolas, that is

of its two opposite vertices, approaches, reaches, and passes through its transition form the elliptic paraboloid; how, having passed through that limit, by its two principal hyperbolas having passed through parabolas into ellipses, and by its principal ellipse having changed in its passage through infinity from imaginary to real, it degenerates into an ellipsoid;—and how, finally, that ellipsoid, by the gradual and simultaneous contraction of all its principal sections, becomes smaller and smaller until at length it vanishes in a point;—and again, during all that transition of the surface itself from species to species, how the corresponding focal conic containing the focus of generation passes in corresponding transition through the successive states of an ellipse, a parabola, an hyperbola, two right lines, the conjugate hyperbola, a parabola, and an ellipse.

For particular positions of the focus directrix and directive axis, as well as for particular values of ϕ, we of course get for any assigned value of m a particular surface of the general species corresponding to that value of m; for instance, for $\phi = 0$ whatever be the position of the focus and directrix, the two directive planes coincide, and we get of course for all values of m a surface of revolution. And here again we see very clearly, from the same enumeration, the cause of the absence of the excepted surfaces; for in that enumeration we saw that all the surfaces generated from a point on the focal *hyperbola* arose from values of m intermediate to 1 and $\cos\phi$, and therefore when these values become equal, as they do when $\phi = 0$, we necessarily lose all the intermediate surfaces. Thus in this case, from $m = \infty$ to $m = 1$ we have an hyperboloid of revolution of *one* sheet, with the focus on its focal *circle*, and from $m = 1$ to $m = 0$ we have an *oblate* ellipsoid of revolution, with the focus also on its focal *circle;* while at the transition value of m, the modulus corresponding indifferently to the value 1 or to the value $\cos\phi$, the resulting surface, besides being a surface of *revolution*, must moreover be at the same time a particular case of the surface corresponding to the former value, that is of an *hyperbolic* paraboloid, and also a particular case of the surface corresponding to the latter value, that is of an *elliptic* paraboloid; and so it is all three at the same time, the *parabolic cylinder* being indifferently an elliptic paraboloid, an hyperbolic paraboloid, and a surface of revolution. The three excepted surfaces of revolution correspond on the contrary to $\phi = 90°$, which from the coincidence of the directive planes always

gives also a surface of revolution; but at that particular value of ϕ the modular method fails in consequence solely of its becoming indeterminate. What we therefore must do in this, as in all such cases, is to find the *limits* to which the resulting surfaces tend for different values of m, when ϕ becomes infinitely near to and ultimately takes that particular value of 90°; and in this point of view we see that the excepted surfaces are, as Professor MacCullagh termed them, the limits to surfaces which *could* be generated modularly. As for the other two species of cylinder, the elliptic and the hyperbolic in their general state, they obviously are but particular cases of the elliptic and hyperbolic paraboloids, and therefore result for the same values of m that give those surfaces themselves, viz. the elliptic cylinder for $m = \cos\phi$ and the hyperbolic for $m = 1$, the particular position of the focus and directrix, with respect to the directive axis which causes a paraboloid to degenerate into a cylinder, being obviously, in the case of the elliptic paraboloid when the plane containing both focus and directrix is *parallel* to the directive axis, and in the case of the hyperbolic paraboloid when it is *perpendicular* to that axis, no normal plane to a focal parabola of either of these surfaces (and the plane of a focus and corresponding directrix is always, as we have seen, a normal plane at the focus to the corresponding focal conic) being ever able to attain the above positions in the *general* state of their respective surfaces.

During the whole transition of m in the general case from ∞ to 0, we met with but one case of a *cone*, where it appeared between $m = 1$ and $m = \cos\phi$ only in its character of being the common asymptotic limit and transition state between the two species of hyperboloid, and with the focus not at its vertex but on one of its focal lines. The cone however, if we take it in all its modifications of form, of a cone in its general state, real, infinitely slender, or imaginary, and of two planes, diverging, coincident, or imaginary, being obviously the limit not only to the two hyperboloids but more generally to all the five surfaces, must consequently enjoy a much wider range as to its capacity of being produced by the modular method; and accordingly we find that when the focus is taken *on* the directrix, and when therefore the resulting surface must obviously be always a cone of some shape or other, with the focus always at its vertex, then as before, in following m through all its transitions of value from ∞ to 0, we get, from $m = \infty$ to $m = 1$, a cone always in the

same dihedral angle between the two directive planes, that is a particular hyperboloid of one sheet placed in the same direction; for $m = 1$, two diverging planes, the directive planes themselves, that is a particular hyperbolic paraboloid; from $m = 1$ to $m = \cos\phi$, another cone, but placed in the supplemental dihedral angle between the directive planes, that is a particular hyperboloid either of one or of two sheets placed in a position different from the first hyperboloid of one sheet; for $m = \cos\phi$, a right line, the directive axis itself, that is, an infinitely slender elliptic paraboloid; and finally, from $m = \cos\phi$ to $m = 0$, a point, the focus itself, that is, an infinitely small evanescent ellipsoid.

All this appears also *à priori* from the peculiar nature of the cone, which is decidedly the most remarkable surface in the modular system: that surface alone, out of all the surfaces of the second order, has its three focal conics all real, and all having a common point at its vertex, two of them being there evanescent ellipses in the transition state between real and imaginary existence, and the third the limit to an hyperbola consisting of two right lines intersecting at that point. Hence, since in every surface of the second order two of the three focals are always modular and the third always umbilicar, we see that as two modular focals pass always through the vertex of a cone, that point must always be not only modular but *doubly* modular, and also that as an umbilicar focal also passes always through the same point, that point must moreover be always not only a modular but also at the same time an umbilicar focus: these two properties are both unique and are not either of them possessed by any point in any other surface of the second order, the cone thus enjoying two very remarkable properties, not either of them possessed by any other surface of the second order, in consequence, as Professor MacCullagh has ingeniously observed, of its being at the same time the asymptotic limit to the two species of hyperboloid, for both of which the focal *hyperbola* is always modular, and for one of which the focal *ellipse* is also always modular while for the other it is always umbilicar; hence in one case the point at which the two focals meet in the limit, that is the vertex of the cone, must be doubly modular, and in the other it must be both modular and umbilicar: the three real dirigent cylinders in a cone corresponding to its three real focals are, obviously, a pair of infinitely slender cylinders, and, two intersecting planes, the infinitely slender cylinder corresponding to the evanescent *modular* ellipse being always

the *internal* axis of the cone, and that corresponding to the evanescent *umbilicar* ellipse being always the *directive* axis, while the two *dirigent planes* corresponding to the focal lines intersect always in the *mean* axis, and lying symmetrically both outside the cone are obviously always so situated that the trace of each on the principal plane of the focal lines forms always with the corresponding focal line, and with the two principal sides of the cone, a harmonic pencil of which the trace and focal line are always one pair of conjugates and the two sides of the cone the other. The names internal and directive, applied by Professor MacCullagh to two of the three axes of a cone, manifestly explain themselves, and the name mean was applied by him to the third in consequence of its being coincident in direction with the mean axis in the two species of hyperboloid of which the cone is the common limit: also that modular ratio of a cone which corresponds to the vertex alone, but not to the focal lines, Professor MacCullagh has called, for the sake of distinction, the *singular* modulus, and that ratio being the limiting *elliptic* modular ratio of an hyperboloid of one sheet is therefore always greater than unity, while the other modular ratio of a cone corresponding to the whole extent of the focal lines he has called for the same reason the *linear* modulus, and that ratio being the limiting *hyperbolic* modular ratio of either of the two hyperboloids is therefore always less than unity: and if moreover in the same surface these two moduli, linear and singular, be denoted respectively by m and n, and that ϕ and $90^\circ - \phi$ be the common inclinations of the directive planes to the principal planes of the focal lines and of the external axes respectively, then in the cone, as in every other surface of the second order, these three quantities, m, n, and ϕ, are always connected by the equation $\frac{\cos^2\phi}{m^2} + \frac{\sin^2\phi}{n^2} = 1$, an equation which we have already noticed as being due to the same author.

Hitherto, in the enumeration of the different surfaces for different values of m, we have all along considered the focus and directrix, however situated, as being always both real; but in order to bring in the imaginary modular focals with their corresponding imaginary dirigent cylinders, as well as to render the enumeration complete, let us now take them both imaginary. In this case, denoting by ϕ the same angle as before, that is, the angle that the directive planes make with the principal plane of the *real* focal conic, and by n the new modular ratio, we see at once, from the same

general equation of connection which holds in all cases $\frac{\cos^2\phi}{m^2} + \frac{\sin^2\phi}{n^2} = 1$, that from $n^2 = 0$ to $n^2 = -\infty$, the surface will be always an ellipsoid; that from $n^2 = +\infty$ to $n^2 = \sin^2\phi$, it will be always an hyperboloid of two sheets; that for n^2 passing through infinity, it will be always an elliptic paraboloid, the limiting and transition surface between the two former; and that, finally, from $n^2 = \sin^2\phi$ to $n^2 = 0$ the surface will be always altogether imaginary.

Let now this method of generation, whose application to curves and to surfaces has been found in the two cases to be so perfectly analogous in its nature, in its exceptions, and in its results, be varied by substituting for the focus, in the case of curves, a circle, and in the case of surfaces, a sphere, of arbitrary radius, and placed in any position; and let it be required to find, in the former case, the curve locus of a point whose tangential distance from the fixed circle is to its distance from the directrix, measured parallel to the directing right line, in a constant ratio, either real or imaginary, and in the latter case, the surface locus of a point whose tangential distance from the fixed sphere is to its distance from the fixed directrix, measured parallel to the fixed directing plane, in a constant ratio, real or imaginary. It will without the least difficulty be seen, and therefore it need not be proved, that, as before, both loci will be always of the second order.

But, moreover, it is also easy to see, *à priori*, that the curve in the former case, whatever be its nature, will always have double contact with the fixed circle, real or imaginary as the case may be, and that the surface in the latter case, whatever be its nature, will also have always double contact, real or imaginary, with the fixed sphere, the chord of double contact in both cases being the directrix, and the two points of contact being therefore, in both, the two points real or imaginary in which that line intersects the circle or sphere; for, in either case, if a point be common either to the curve and circle or to the surface and sphere, then, since the tangential distance of that point from the circle or sphere is nothing, so must the distance of the same point from the directrix be also nothing, and that whether the given constant ratio be real or imaginary: hence the two particular points on the circle or sphere, in which either is met by the directrix, satisfy the conditions of the question, and are therefore always on the curve or surface locus of the moveable point, while no other point on either, as unable to fulfil

the conditions, can ever in either case belong to that locus. And again, when the constant modular ratio is *real,* no point of either curve or surface can ever lie *within* the circle or sphere, for in such case the tangential distance of every *internal* point would be imaginary, while the distance from the directrix of *every* point, whatever be its position, measured in any direction, must always be real. And when, on the contrary, the constant modular ratio is *imaginary,* the circle or sphere and the directrix being both real, then, conversely, for plainly the same reason, no point on the locus curve or surface can lie *outside* the circle or sphere: from which last we see, since a curve or surface lying altogether within a circle or sphere must be necessarily closed, that an imaginary modular ratio could out of all the curves and surfaces of the second order produce only the ellipse and the ellipsoid.

This property of the locus being general and independent of the arbitrary magnitude and position of the circle or sphere, as well as of the arbitrary magnitude, real or imaginary, of the modular ratio, we may obviously combine it in both cases with the results of the modular generation, by conceiving the circle in the one case and the sphere in the other to dwindle into points fixed, but arbitrary in position, and real or imaginary, as the case may be; and we thus arrive at the following important and general properties, which have the closest analogy to each other in the different cases of curves and of surfaces of the second order.

Each focus, real or imaginary, of a curve of the second order is the evanescent limit to a circle having double imaginary contact with the curve, the chord of double contact, obviously in all cases perpendicular to a principal axis of the curve, being in that particular limiting case the directrix of the curve which corresponds to that focus*; and every point on a *modular*

* This first property, which has been long known, though not, so far as the author is aware, been made much use of, may be established in another manner altogether different from the above, by considering the focus of a conic as the umbilicus of an infinitely flat surface of the second order, and by applying the obvious principle that the projection from any point upon any plane of every plane section of a surface of the second order will be a conic which will always have double contact, real or imaginary, with the contour of the projection of the surface itself, the real chord of which will be always the projection of the line in which the plane of the section intersects the plane of contact of the cone which from the point envelopes the surface. Hence the infinitely small circle at the umbilicus of a surface of the second order will be always projected into an infinitely small conic having double contact with the outline of the projected surface itself, and

focal conic, real or imaginary, of a surface of the second order is the evanescent limit to a sphere having double imaginary contact with the surface, the chord of double contact, obviously in all cases perpendicular to a principal plane of the surface, being in this particular and limiting case the directrix of the surface which corresponds to that focus, and the peculiar and distinguishing nature of the double contact in the case of this the modular species of focal conic, whether real or imaginary, being such, that even when in the general state of the touching sphere the double contact with the surface changes from imaginary to real, still the two plane curves of intersection, which always accompany the double contact of any two surfaces whatever of the second order,* remain in this (the modular) case always imaginary.

Now there exists an essential and important difference in the nature of the double contact of the three systems of spheres which can thus touch every surface of the second order. In all cases the chord of contact of every individual sphere must be parallel to an axis of the surface; for, drawing through the line of intersection of the common tangent planes at the two points of contact a plane bisecting the chord of double contact, that plane, since the tangent planes and the chord belong to the surface, must be a diametral plane of the surface and the diametral plane conjugate to the chord, and since they belong to the sphere, it must intersect the chord at right angles and be therefore a principal plane of the surface, and the chord therefore a principal chord parallel to an axis. But for the system of spheres of which the chords are parallel to the *directive* axis of the surface, and therefore perpendicular to the plane of the single *non-modular* focal conic, the two accompanying plane curves of intersection, obviously always both circles, as in the three cases belonging to spheres, will be always both real when the contact is itself real, and will very often one or both be also real even

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therefore when the surface flattens down into a plane, in which case the projection of the little circle will be the little circle itself, and the projection of the surface will be its boundary curve, the focus will be the evanescent limit to a circle having double contact with that curve.

* For, through the common chord of double contact and through any point on the curve of intersection of the surfaces, let a plane be drawn intersecting the surfaces in a pair of conics and the two common tangent planes in a pair of common tangents to these conics; the two conics will then have three points and two tangents in common, and will therefore coincide with each other. This elegant and simple proof is due to M. Poncelet.

when the contact itself is imaginary; while for the other two systems of spheres, that is for those whose chords of double contact are perpendicular to the planes of the two *modular* focal conics, the two circles of intersection will both be always imaginary, so that in both these latter cases the two points of contact, even when they are real, are always isolated, and the spheres of contact have none of them in either case any other point in common with the surface. All this is obvious from the known fact that the line of intersection of the planes of every pair of subcontrary real circular sections in any surface of the second order is in all cases parallel to the directive axis of that surface.

The consideration of this difference enables us to shew readily that every point on the *umbilicar* focal conic, whether real or imaginary, of a surface of the second order possesses the same important property of being the evanescent limit, real or imaginary, to a sphere having double imaginary contact with the surface. To see this, having considered in the former case the modular method, let us now consider Mr. Salmon's method of generating surfaces of the second order, a method which has also an obvious analogy in the generation of curves of that order, and which was suggested to him by the consideration that but two of the three focal conics were ever connected with the modular method of generation in any surface of the second order.

He has sought for the locus of a point in space, the square of whose distance from a fixed point is to the rectangle under its distances from two fixed planes in a constant ratio, and of this locus (obviously always a surface of the second order and always symmetrical on both sides of the plane drawn through the point perpendicular to the line of intersection of the planes) he has discussed the properties by considering the reciprocal surface to a sphere of arbitrary radius placed with its centre at the fixed point: in this reciprocation he has used a simple and obvious but very important property of the sphere, due also to himself, viz., "the distances of any two points assumed arbitrarily in space from the centre of any sphere of magnitude and position also arbitrary, have always the same ratio as their respective distances each from the polar plane of the other with respect to that surface." * The

* As the method of reciprocal polars, in general so powerful in all inquiries relating to position and figure, is far from being equally so in geometry of magnitude, it was in some degree to remedy this known and felt imperfection in the particular but far most important and practicable case, when a sphere is the particular surface of the second order employed

introduction of this property obviously reduces the finding of the surface reciprocal to the above locus to the determination of the surface envelope of a plane, the rectangle under whose distances from two fixed points is to the rectangle under the distances of those points themselves from a third fixed point in a constant ratio, that is, of a plane the rectangle under whose distances from two fixed points is constant. Hence, the reciprocal being a *prolate* surface of revolution of the second order, that is a surface generated by the revolution of the meridian round its *primary* axis, the centre of the reciprocating sphere, that is the fixed point in this method of generation, must be on the *umbilicar* focal conic of the locus itself. For this reason, as well as for the sake of distinguishing it from the modular method, Mr. Salmon's method of generating surfaces of the second order has been called the *umbilicar method* of generation.*

Here also, as in the modular method of generation, it is obvious *à priori*, that whatever be the nature of the generated surface, it must always be symmetrical on both sides of the plane passing through the fixed point (which from analogy has also been called a focus), and perpendicular to the intersection of the two fixed planes (which intersection has for the same reason been called a directrix). And here again, as in the other method, it is also evident *à priori* (as Mr. Salmon has noticed), that all sections of the surface, whatever be its nature, by planes parallel to either of the two fixed or directing planes, will be always circles; for, taking any one individual plane of either of these two systems of parallel planes, the curve in which it intersects the generated surface (since all points of that curve are equidistant from one of the two fixed planes)

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in the reciprocation, that Mr. Salmon investigated and proposed the above theorem, which from its efficiency may certainly be considered as a principle fundamental in the application of that powerful method in all inquiries relating to magnitude.

* By taking the two directing planes coincident with each other, we obviously get immediately by this method of generation the three species of surface of revolution which the modular method failed in supplying, viz. the three produced by the revolution of the three species of conic round their *primary* axes; while, on the contrary, the other three species which were got readily by the modular method, cannot by any variation of the circumstances be obtained by this method at all: hence, as in the former case the name *modular* was suggested in preference to *oblate* for the purpose of distinguishing as a class the whole three species, so in the present case, for the same reason, the term *prolate* being applicable only to one of three species obtained by the umbilicar method, it would be perhaps convenient to distinguish as a class the whole three at once by the common name of *umbilicar* surfaces of revolution.

is the intersection with that plane of the surface locus of points in space, the squares of whose distances from the fixed point are to their simple distances from the other fixed plane in a constant ratio, and is therefore always a circle, since the surface locus of such a system of points in space is always a sphere.

Hence, in Mr. Salmon's as in Professor MacCullagh's method of generation, the generated surface being always of the second order, the focus lies always in a principal plane of the surface, and the directrix is always perpendicular to the same plane: but here, differing from the modular case, the principal plane in which the former lies, and to which the latter is perpendicular, is always that perpendicular to the *directive* axis of the surface, and hence also in Mr. Salmon's method the two fixed directing planes are always parallel to the two cyclic or directive planes of the generated surface. Moreover, as in the modular method Professor MacCullagh has shewn, so in this the umbilicar method Mr. Salmon has shewn, that the focus need not be a unique point nor the directrix a unique right line, but that there always exists an infinite number of foci all in the same principal plane, and also an infinite number of corresponding directrices all perpendicular to the same plane, by which with the same umbilicar ratio and the same directive planes the same surface may be generated: he has also shewn, that as in the former case the plane curve locus of the whole system of foci in the principal plane in which they all lie is always of the second order and is no other than the *umbilicar* focal conic of the generated surface, and that the cylinder locus of the whole system of corresponding directrices is also always of the second order, and is always connected with the corresponding focal conic by precisely the same relation as in the modular case, viz. that the conic and cylinder are always polars reciprocal to each other with respect to the surface, each individual directrix being as before the polar of the line which at the corresponding focus is a tangent to the focal conic, and the plane containing at once focus and directrix being always perpendicular to that focal tangent. This cylinder, following the analogy of the modular case, has been called with respect to the surface the *umbilicar dirigent cylinder*, and its base, obviously the polar reciprocal of the umbilicar focal conic with respect to the principal section of the surface in their common principal plane, has for the same reason been called the *umbilicar dirigent conic.*

In the ellipsoid, the elliptic paraboloid, and the hyperboloid of two sheets, as well as in the particular cases of the two latter, the cone, and the elliptic cylinder, one real focal conic, with its corresponding real dirigent cylinder and real dirigent conic, is always modular and the other always umbilicar; but in the hyperboloid of one sheet and the hyperbolic paraboloid, as also in two of the three particular cases of the latter, viz. the hyperbolic and the parabolic cylinders, neither of the two real focal conics, and therefore neither of the accompanying dirigent cylinders or dirigent conics, is ever umbilicar. An umbilicar may (as we said before) be always distinguished from a modular focal conic by the fact of its always intersecting the surface: this intersection always takes place at the points of the surface, real or imaginary, where the tangent planes are parallel to the directive planes, that is at the umbilici: and that such should be the case appears *à priori* from the nature of the umbilicar method of generation, for, whenever the focus is taken on either of the two directing planes, then must the resulting surface, whatever be its nature, pass always through that point, and there touch that plane, for the square of the distance of that particular point on the plane from the focus and the rectangle under its two perpendicular distances from the two directing planes being both at the same time nothing, that point satisfies the conditions of the question and is therefore on the locus, while, no other point in the plane possessing the same property, the locus can never meet it again and must therefore touch the plane at that point. On the contrary, as we have shewn *à priori*, the modular focal conic, whether real or imaginary, never meets its surface except in one particular case, viz. at the vertex of the cone, a case already considered and included, as we have seen, under the general property that in every surface of the second order two of the three focal conics are always modular, and the third is always umbilicar, the single umbilicar focal, real or imaginary, lying always in the principal plane of the surface perpendicular to its directive axis, and the two modular focals lying always in the two principal planes which intersect in that axis.

As in the modular so in the umbilicar method of generation, when we are given the umbilicar ratio and the common inclination of the directive planes to either of the two modular principal planes, the species of the resulting surface may, to a certain extent, be always determined *à priori* independently of the arbitrary positions of the focus

and directrix. To see this, as in the other case so now, denoting by μ^2 the umbilicar ratio and by ϕ the inclination, let the principle of the umbilicar generation be thrown by the reader into algebraical language, the focus, directrix, and directing planes being considered as all real, but as taken quite arbitrarily and not in any particular positions with respect to each other, which, as before, would of course give only particular surfaces, and in the resulting equation let μ^2 be followed in all its successive transitions of value from ∞ to 0; then will it be immediately evident to mere inspection, that, from $\mu = \infty$ to $\mu = \sec\phi$, the surface will be always an hyperboloid of *two* sheets with the focus of generation on its focal *ellipse*, ϕ being the common inclination of the directive planes to the principal plane of the surface containing the two imaginary axes;—that, for $\mu = \sec\phi$, the surface and focal will both simultaneously reach their limiting and transition states, the former degenerating into an *elliptic* paraboloid, and the latter into the intersecting *parabola*;—that, from $\mu = \sec\phi$ to $\mu = 0$, the surface having passed from an hyperboloid of two sheets through an elliptic paraboloid becomes and remains an *ellipsoid*, and the focal having passed from an ellipse through a parabola becomes and remains an *hyperbola*;—and that, finally, for $\mu = 0$, the ellipsoid dwindles into a point and becomes evanescent.

Hence we see that with a real focus and directrix, the only kind of course that are practically available, we can get by the umbilicar method of generation only the three convex surfaces of the second order in their general state, but not either of the two non-convex surfaces in their general, and therefore also not in any of their particular states, an imperfection which of course renders this method, as compared with the modular, of very inferior value. By giving particular values to ϕ, as well as particular positions to the focus and directrix, we get of course for the different values of μ, corresponding particular cases of the same three surfaces respectively. Of these, by taking the point *on* the directrix, we get the three corresponding species of cone with its vertex at the focus, viz. from $\mu = \infty$ to $\mu = \sec\phi$, the cone in its general state, the limit to an hyperboloid of two sheets, for $\mu = \sec\phi$, an infinitely slender cylinder or evanescent elliptic paraboloid; and from $\mu = \sec\phi$ to $\mu = 0$ a point or evanescent ellipsoid. And again, by taking $\phi = 0$, that is by taking the two directive planes coinciding with each other, we get for the same values of μ the three corresponding surfaces of revolution which are those missed in the modular method, viz. those

generated by the revolution of the three conics in their general state round their *primary* axes.*

Substituting now in the umbilicar, as before in the modular method of generation, for the fixed point, a fixed sphere of arbitrary radius, and placed in any position, and seeking for the locus of a point in space, the square of whose tangential distance from the fixed sphere is to the rectangle under its perpendicular distances from two fixed planes in a constant ratio, there will be not the least difficulty in seeing that, as in the former case, the resulting surface will be always of the second order.

Moreover, in this case also it is easy to see, *à priori*, that the locus, whatever be its nature, will always have double contact with the sphere, real or imaginary as the case may

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* With respect to the relative advantages of these two methods of generating the surfaces of the second order, the modular and the umbilicar, which have been thus brought before the reader's notice for the purpose of establishing from them a general focal principle in all surfaces of the second order, we may observe that towards producing the five principal surfaces in their general state the former is by very much the more powerful of the two, since by it we get the whole five in their general and in almost all their particular states, while from the latter we can only obtain the three convex surfaces in their general state with the cone in its varieties of form which correspond to the same three surfaces, but not either of the undevelopable rule surfaces in their general or in any of their particular states, nor even any of the three cylinders. It is only in producing the six surfaces of revolution in their general and in their two transition states that they seem to possess any like equal power, and in this respect the analogy between them is very striking; for while the three surfaces generated by the revolution of the three species of conic in this general state round their primary axes may be at once obtained by one of these methods, but not at all by the other, and while conversely the three generated by the revolution of the three conics round their secondary axes may be at once obtained by the latter, but not at all by the former, the sphere, on the one hand, which is the common limit and transition state between the two species of ellipsoid of revolution, is equally excluded from both, and on the other hand, the right circular cone in its general state, which is the common limit and transition state between the two species of hyperboloid of revolution, is equally obtainable by either.

Again, with respect to the analogy existing between these two methods of generating surfaces and the corresponding two methods of generating curves, we may observe that both with respect to the included and to the failing cases it seems very complete; for as in the generation of curves the three principal curves of the second order in their general state may be got by both methods, so in the generation of surfaces the three principal convex surfaces of the second order in their general state may be obtained by both methods; and as in the former case two particular curves, viz. the circle and two parallel right lines, may in a certain sense be considered as excluded from both methods for curves, so in the latter case the three particular surfaces of revolution which have the very same two curves for meridians, viz. the sphere, the right circular cylinder, and two parallel planes, may in the very same sense be considered as excluded from both methods for surfaces.

be, and that the chord of double contact will, as before, be the directrix, that is in this case the intersection of the two planes. For, while it can have no other point in common with the sphere, it must obviously, whatever be its nature, pass always through the two circles in which that surface intersects the two planes, as appears at once from considering that all points on the locus for which the tangential distance from the sphere is nothing, must be also such that the rectangle under their distances from the planes must also vanish, and the twofold property being obviously possessed by no other point or points on the sphere except those in which it is intersected by one or other of the planes. Hence the locus, passing always through two plane curves on the sphere, must always have double contact with that surface, the points of contact being the two points, real or imaginary, in which the two circles intersect, and the chord of double contact being therefore the right line joining these two points, that is, the line in which the two fixed planes intersect.

Different however from the former case, part of the surface may here lie within the sphere and part without it, or it may lie wholly within or wholly without, as the case may be, and that whether the constant umbilicar ratio be positive or negative; for, the rectangle under the perpendiculars on the fixed planes passing through nothing, changes sign in the transition of the moveable point from the region included in any one of the four dihedral angles made by those two planes into the region included in the supplemental dihedral angle, and therefore its tangential distance from the sphere changes in the same transition from real to imaginary, or *vice versâ*, as the case may be. Hence, according as the sphere lies wholly in one of these regions, or wholly in the other, or partly in one and partly in the other, so will the generated surface lie either wholly within or wholly without the sphere, or it will lie partly within that surface and partly without it.

This general property of the locus being, as in the former case, independent of the arbitrary magnitude and position of the fixed sphere, as well as of the arbitrary magnitude positive or negative of the umbilicar ratio, we may obviously combine it with the results of the umbilicar generation, as we did before the analogous property in the case of the modular generation; and thus, conceiving the sphere to dwindle into a point fixed but arbitrary in position and real or imaginary as the case may be, we arrive at the important property that every point on the *umbilicar* focal conic, real

or imaginary, of any surface of the second order is also the evanescent limit to a sphere having double imaginary contact with the surface; the chord of double contact, obviously in all cases perpendicular to a principal plane of the surface, being in this particular and limiting case the directrix of the surface, which corresponds to that point as focus, and the peculiar and distinguishing nature of the double contact in the case of this the umbilicar species of focal conic, whether real or imaginary, being such that whenever in the general state of the touching sphere the double contact changes from imaginary to real, then always, and often even while the contact itself remains imaginary, are the two accompanying plane curves of intersection also both real.

The principle is therefore general, and holds in all surfaces of the second order for the three focal conics.* It might have been established by a single analytical solution for the whole three at once by taking the equation of a sphere having double contact with the surface, and from that equation shewing that the coordinates of the centre of every such sphere, obviously in all cases lying in a principal plane of the surface, satisfy the equation of the focal conic in that plane when the radius of the sphere is put equal to nothing. But in this, as indeed in every inquiry of the same nature, a geometrical solution, whenever it can be obtained, possesses the advantage of shewing not only that a property or principle is true, but also why it is so.

But by whatever means established, the principle itself however seems an important one, and not only expresses a most perfect and intimate analogy between all the foci real or imaginary of curves, and all the focal conics real or imaginary of surfaces, of the second order, the former being in both cases the evanescent limits to circles having double contact with their curve, and the latter in both cases the loci of the evanescent limits to spheres having double contact with their surface; but moreover it also furnishes, in the first place, a general and uniform method, simple and at the same time efficient, for investigating with ease and rapidity all the

* The above general principle, to which we attach some importance as an instrument of direct investigation and which we now proceed to illustrate, it is to be remembered has been established in the present paper by setting out from the fundamental geometrical definition of surfaces of the second order and of those points in them which are called foci, and not by assuming any subsequently derived properties of focal conics considered in any other light than as merely the loci of the three different and distinct systems of focal points which in every surface of the second order may be employed in the generation of that surface.

focal properties in surfaces of the second order; and in the next place a powerful and general test, at once certain and easy of application, for ascertaining in unknown or undetermined cases whether a point, real or imaginary, which enjoys known properties with respect to any surface of the second order, or which occupies an important position in its generation, be or be not upon one of the focal conics of that surface; and moreover, if it be upon a focal conic, whether that conic, real or imaginary, be a modular or an umbilicar focal. For, every thing else remaining unaltered, we have but for the point to substitute a sphere, and if then all hold as before, lines to the point being replaced by tangents to the sphere, the point will or will not be upon a focal conic, according as the sphere has or has not double contact with the surface: and again, if it be on a focal conic, that conic, whether real or imaginary, will be modular or umbilicar according as the two circles of intersection which accompany the double contact of the sphere and surface are imaginary or real.

For the case of curves of the second order the test is, of course, the same for ascertaining whether a point under similar circumstances be or be not a focus of the curve, for, having replaced the point by a circle, we have but to ascertain whether the substituted circle have or have not double contact with the modified curve. But before however we can with safety apply this method in either case to any particular instance, we must previously have determined that the substitution of the sphere or circle does not affect the essential features of the question, or cause the surface or curve before of the second order to cease being any longer of that order; for, if such be the case, it is of course not applicable, but with this necessary precaution the method seems otherwise safe and easy of application. We will now first give a few examples illustrative of the rapid facility with which all the focal properties in surfaces of the second order may be developed by means of this general principle, and then apply it as a test to the very interesting example afforded by M. Jacobi's method of generating surfaces of the second order.

If any two surfaces whatever of the second order have double contact, whether real or imaginary, the chord of double contact, always real when the surfaces themselves are real, will obviously, whether real or imaginary, have always the same polar with respect to both surfaces, viz. the intersection, real or imaginary, of the two common tangent planes at the two points of contact.

Let now one of these two surfaces be a sphere: both these common polars with respect to that surface, and therefore with respect to the other whatever be its nature, will then obviously be always perpendicular, each to the plane passing through the centre of the sphere and containing the other: and finally let the sphere dwindle into a point, real or imaginary; the polar of the chord of double imaginary contact with respect to it, and therefore with respect to the other surface, besides being still always perpendicular to the plane containing that chord and passing through the centre of the evanescent sphere, will moreover, whether real or imaginary, in this limiting case obviously pass always through that point itself.

Hence we see that the polar with respect to any surface whatever of the second order of every directrix of that surface, whether modular or umbilicar, passes always through the corresponding focus, and is always perpendicular to the plane containing that point and directrix.

If any two spheres in position infinitely near to each other and in magnitude differing infinitely little from each other have both double contact with any surface of the second order, then, since every two spheres, whatever be their relative position or magnitude, intersect always in a circle, the plane of which is always perpendicular to the line joining their centres, when these two spheres come together and ultimately coincide, their circle of ultimate intersection will obviously pass always through the two common points of double contact, while the right line in which their centres move into each other just before they come together and ultimately coincide, will, as in every other so of course in that limiting case, be always perpendicular to the plane of their ultimate intersection.

Hence, when these two spheres both dwindle into points infinitely near to each other, and when therefore the line joining their centres is a tangent to a focal conic of the surface with which they both have double contact, we see that the plane which connects any focus of a surface of the second order, whether modular or umbilicar, with its corresponding directrix is always perpendicular to the focal tangent at that point.

Combining this last with the immediately preceding property, viz. that the perpendicular to that plane which passes through the focus is always the polar with respect to the surface of the corresponding directrix, we see that at every focus of a surface of the second order, modular or umbilicar,

the focal tangent is always the polar with respect to the surface of the directrix corresponding to that focus, and is therefore always the polar with respect to the principal section of the surface in the corresponding plane of the foot of the directrix; so that hence also, each focal conic, with its corresponding dirigent cylinder with respect to the surface, and with its corresponding dirigent conic with respect to the principal curve, are in all cases connected by the relation of being always polars reciprocal to each other. These are the relations which we stated were shewn by Professor MacCullagh and by Mr. Salmon to exist between the focal and its corresponding dirigent in their respective methods of generating surfaces of the second order, and we now see that they follow in both cases simply and readily from a single geometrical property common equally to both species of focal conic, whether real or imaginary, in every surface of the second order.

When any two surfaces whatever of the second order have double contact, real or imaginary, then will every point assumed arbitrarily on the common chord have always the same polar plane with respect to both surfaces, viz. that passing through the line of intersection of the two common tangent planes, and through the point of harmonic section of the chord conjugate to the assumed point with respect to the extremities of that chord.

If now one of the two surfaces be a sphere, its polar plane will be always perpendicular to the line joining the point with its centre; and if finally the sphere dwindle into a point, then also will the polar plane pass always through that point.

Hence, as in a curve of the second order the polar of every point upon a directrix of the curve passes always through the corresponding focus and is always perpendicular to the line joining the point and focus, so in any surface of the second order the polar plane of every point upon a directrix of the surface passes always through the corresponding focal tangent, and is always perpendicular to the line joining the point and focus.

From this we immediately derive a simple geometrical construction for finding the pole with respect to any surface of the second order of any plane which touches a focal conic of that surface, modular or umbilicar, and, conversely, for finding the polar plane of any point on a directrix of the surface of either species; for in the former case we have but to erect a perpendicular to the plane at its point of

contact with the focal conic, and then that perpendicular will meet the corresponding directrix in the pole required; and in the latter case we have but to connect the point on the directrix with the corresponding focus, and then the plane drawn through that point perpendicular to the joining line will be the polar plane required.

From these again we derive at once the following geometrical constructions for finding the polar plane of any point whatever with respect to a surface of the second order, and, conversely, for finding the pole of any plane whatever with respect to the surface; for in the former case drawing through the given point any three tangent planes whatever to a focal conic of the surface and erecting perpendiculars at their points of contact to meet the corresponding directrices, then will the plane passing through the three points of meeting be obviously the polar plane required; and in the latter case taking the three points in which the given plane intersects any three directrices of the surface, connecting these points with the corresponding foci, and drawing through each focus a plane perpendicular to the joining line, then will the point in which these three planes intersect be obviously the pole required.

Since in the latter case we are at liberty to select any three directrices we please, let us choose as two of them those corresponding to the two points, real or imaginary, in which the given plane intersects either of the focal conics; then in that case, whatever be the third, the two lines joining the points in which the given plane intersects these two particular directrices with the corresponding foci will always lie in that plane itself, and the two planes drawn through the foci perpendicular to these joining lines will therefore be always both perpendicular to that plane and will pass through the two focal tangents at those foci respectively. Hence the line of intersection of these two perpendicular planes, a line which, according to the above general construction, must always contain, and therefore always pass through the pole of the given plane, will be always a perpendicular to that plane, and will always pass through the real intersection of the two focal tangents at the points, real or imaginary, where it meets the focal conic, that is, through the pole with respect to that focal conic of the line in which the given plane intersects the principal plane of that curve.

Hence, from the same simple geometrical principle, common alike to the two species of focal conic and to the imaginary as well as to the real conics of each species, we

arrive at the important theorem in surfaces of the second order, that if from any point assumed arbitrarily in space a perpendicular be let fall upon its polar plane with respect to any surface of the second order, then will the point and line of meeting, of the perpendicular and plane respectively, with each principal plane of the surface be always pole and polar to each other with respect to the focal conic, whatever be its nature, in that principal plane.

Of a particular case of this important theorem—viz. "the point and line in which the normal and the tangent plane at every point of any surface of the second order meet respectively each principal plane of the surface will be always pole and polar to each other with respect to the focal conic in that principal plane"—a very elegant geometrical solution has been given by Professor MacCullagh; but of the theorem itself in its general state, a geometrical solution (so far as the author is aware) has not hitherto been given by any writer. Of its great importance in the geometrical theory of confocal surfaces of the second order, we need only say that it is possible, setting out from it as our fundamental principle and making it the basis of our subsequent investigation, to deduce synthetically, in regular and connected order, almost all the known properties of that highly interesting and instructive class of systems of surfaces of the second order.*

[*To be continued.*]

Trinity College, Dublin, February, 1847.

ON ASYMPTOTIC PLANES AND ASYMPTOTIC SURFACES.

By William Walton, M.A., Trinity College.

A plane touching a surface at an infinite distance and passing within a finite distance from the origin of coordinates, may be called an asymptotic plane. Sometimes a surface admits of only one such asymptotic plane, and sometimes of an infinite number of them consecutively intersecting: in the latter case the consecutive intersections will constitute a developable surface touching the proposed surface asymp-

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* That this assertion is true, in the case of at least one class of properties, will be seen from the continuation of the author's paper on the Principal Axes of Bodies, in which the above theorem is made the basis of the whole development of the general laws which govern the distribution in space of the principal axes of every solid body; but in the present paper our necessary limits prevent of course the possibility of even an attempt at illustration, or of even a moment's digression, on the subject of confocal surfaces.

totically. The equations to the asymptotic developable surfaces of surfaces of the second order, namely their asymptotic cones, are given in the ordinary treatises on Solid Geometry, where they are obtained by particular considerations. I am not aware that any general method of finding the asymptotic planes and asymptotic developable surfaces has been laid down by geometricians. The process however for the discovery of the asymptotes of plane curves* may easily be extended so as to embrace the determination of the position of asymptotic planes of surfaces; and then, by the ordinary method of finding the envelop of a family of surfaces, we may investigate the asymptotic developable surfaces.

Let the equation to any surface be

$$\phi_n(x, y, z) + \phi_{n-1}(x, y, z) + \phi_{n-2}(x, y, z) + \ldots = 0 \ldots\ldots (\text{I}),$$

the symbol $\phi_s(x, y, z)$ denoting generally a homogeneous function of s dimensions in x, y, z.

Let the equations to a line passing through a point x, y, z, of the surface be

$$\frac{x - x'}{\lambda} = \frac{y - y'}{\mu} = \frac{z - z'}{\nu} = r \ldots\ldots\ldots\ldots (\text{II}).$$

Then, eliminating x, y, z, from (I) and (II), we have

$$0 = r^n \bar{\phi}_n + r^{n-1} \bar{\phi}_{n-1} + r^{n-2} \bar{\phi}_{n-2} + \ldots\ldots,$$

where $\bar{\phi}_s$ denotes $\phi_s\left(\lambda + \frac{x'}{r}, \mu + \frac{y'}{r}, \nu + \frac{z'}{r}\right)$, and therefore, writing ϕ_s instead of $\phi_s(\lambda, \mu, \nu)$, and putting

$$x' \frac{d}{d\lambda} + y' \frac{d}{d\mu} + z' \frac{d}{d\nu} = D,$$

we have $r^n . e^{\frac{1}{r}D} \phi_n + r^{n-1} . e^{\frac{1}{r}D} \phi_{n-1} + r^{n-2} . e^{\frac{1}{r}D} \phi_{n-2} + \ldots = 0$,
whence

$$0 = r^n \phi_n + r^{n-1}(D\phi_n + \phi_{n-1}) + r^{n-2}\left(\frac{1}{1.2} D^2\phi_n + D\phi_{n-1} + \phi_{n-2}\right)$$
$$+ r^{n-3}\left(\frac{1}{1.2.3} D^3\phi_n + \frac{1}{1.2} D^2\phi_{n-1} + D\phi_{n-2} + \phi_{n-3}\right)$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{III}).$$

Suppose that the line (II) is tangential to the surface (I):

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* See a paper on the Asymptotes of Plane Curves, by Gregory, in the *Cambridge Mathematical Journal* for November, 1843.

then, for any proposed system of values of $\lambda, \mu, \nu, x', y', z'$, each of the coordinates x, y, z, must have at least two equal values, and therefore r must have at least two equal values. We have therefore from (III), by differentiating it with regard to r,

$$0 = nr^{n-1}\phi_n + (n-1)r^{n-2}(D\phi_n + \phi_{n-1}) + (n-2)r^{n-3}\left(\frac{1}{1.2}D^2\phi_n + D\phi_{n-1} + \phi_{n-2}\right)$$

$$+ (n-3)r^{n-4}\left(\frac{1}{1.2.3}D^3\phi_n + \frac{1}{1.2}D^2\phi_{n-1} + D\phi_{n-2} + \phi_{n-3}\right)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{IV}).$$

Multiplying (III) by n and (IV) by r, and subtracting the latter of the resulting equations from the former, we get

$$0 = r^{n-1}(D\phi_n + \phi_{n-1}) + 2r^{n-2}\left(\frac{1}{1.2}D^2\phi_n + D\phi_{n-1} + \phi_{n-2}\right)$$

$$+ 3r^{n-3}\left(\frac{1}{1.2.3}D^3\phi_n + \frac{1}{1.2}D^2\phi_{n-1} + D\phi_{n-2} + \phi_{n-3}\right)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{V}).$$

Suppose now that the tangent line becomes an asymptote; then x', y', z', being finite and r infinite, we have from (III), first dividing by r^n and then putting $r = \infty$,

$$\phi_n = 0 \ldots\ldots\ldots\ldots\ldots (\text{VI}).$$

In like manner, from (V) we get

$$D\phi_n + \phi_{n-1} = 0 \ldots\ldots\ldots\ldots (\text{VII}).$$

The equation (VII) which, written at length, is equivalent to

$$x' \frac{d\phi_n}{d\lambda} + y' \frac{d\phi_n}{d\mu} + z' \frac{d\phi_n}{d\nu} + \phi_{n-1} = 0,$$

represents generally a plane involving the parameters λ, μ, ν, these parameters being subject to the condition (VI).

Sometimes the condition (VI) involves a relation among the parameters λ, μ, ν, which causes these quantities to disappear entirely from the equation (VII). In such cases the equation (VII) will represent a single asymptotic plane. In other cases an envelop of the family of planes represented by (VII) may be obtained, the parameters being supposed to vary in accordance with the equation (VI); this envelop will be the asymptotic developable surface of the proposed surface.

In the above considerations we have supposed that, while $\phi_n = 0$, the quantities

$$\frac{d\phi_n}{d\lambda}, \quad \frac{d\phi_n}{d\mu}, \quad \frac{d\phi_n}{d\nu},$$

are not all of them zero. Should this be the case, then, by virtue of (VII), ϕ_{n-1} must also be zero; and the relation between λ, μ, ν, involved in the equation $\phi_n = 0$, will render nugatory the equation (VII). We may readily extricate ourselves from this difficulty by reverting to the equation (V).

Suppose that, by virtue of the equation (VI), the coefficients of r^{n-1}, r^{n-2}, r^{n-3}. . . . . r^{n-t+1}, in the equation (V), all vanish identically, the values of x', y', z', remaining general; then, dividing by r^{n-t}, and afterwards putting $r = \infty$, we obtain the following equation in x', y', z', and the parameters λ, μ, ν,

$$D^t\phi_n + tD^{t-1}\phi_{n-1} + t(t-1)\,D^{t-2}\phi_{n-2} + t(t-1)\,(t-2)\,D^{t-3}\phi_{n-3} + \ldots$$
$$+\, t(t-1)\,(t-2)\ldots 32.1.D^0\phi_{n-t} = 0. \ldots (\text{VIII}).$$

The equation (VIII) will represent generally t asymptotic planes parallel to each other, as we shall presently shew, their parameters λ, μ, ν, being subject to the condition (VI). Thus the envelop of (VIII) will generally be an asymptotic surface consisting of t developable sheets. Sometimes the introduction of the condition (VI) into the equation (VIII) will cause λ, μ, ν, to disappear: in such cases there will be simply t single asymptotic planes.

We proceed to shew that the equation (VIII) does really represent a system of parallel planes.

Since $\phi_n = 0$ involves a relation among the quantities λ, μ, ν, which establishes the equations

$$D\phi_n = 0, \quad D^2\phi_n = 0, \quad D^3\phi_n = 0 \ldots\ldots D^{t-1}\phi_n = 0,$$
$$\phi_{n-1} = 0, \quad D\phi_{n-1} = 0, \quad D^2\phi_{n-1} = 0, \quad D^3\phi_{n-1} = 0, \ldots D^{t-2}\phi_{n-1} = 0,$$
$$\phi_{n-2} = 0, \quad D\phi_{n-2} = 0, \quad D^2\phi_{n-2} = 0, \quad D^3\phi_{n-2} = 0, \ldots D^{t-3}\phi_{n-2} = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\phi_{n-t} = 0, \quad D^1\phi_{n-t} = 0,$$
$$D^0\phi_{n-t+1} = 0,$$

it is plain that the relation may be denoted by $v^t = 0$, and that

$$\phi_n = k_n v^t, \quad \phi_{n-1} = k_{n-1}v^{t-1}, \quad \phi_{n-2} = k_{n-2}v^{t-2}, \ldots \phi_{n-t+1} = k_{n-t+1}v,$$

$k_n, k_{n-1}, k_{n-2}, \ldots k_{n-t+1}$, representing certain functions of λ, μ, ν, of which at any rate all do not involve v as a factor.

Then, when $v = 0$, L, M, N, denoting each of them some function of λ, μ, ν, and P representing the expression

$$Lx' + My' + Nz',$$

we have

$$\begin{aligned} D^t\phi_n &= 1.2.3.\ldots t.k_n.P^t, \\ D^{t-1}\phi_{n-1} &= 1.2.3.\ldots(t-1)\,k_{n-1}P^{t-1}, \\ D^{t-2}\phi_{n-2} &= 1.2.3.\ldots(t-2)\,k_{n-2}P^{t-2}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ D^0\phi_{n-t} &= 1.k_{n-t}.P^0. \end{aligned}$$

Hence the equation (VIII) becomes

$$k_nP^t + k_{n-1}P^{t-1} + k_{n-2}P^{t-2} + \ldots . + k_{n-t}P^0 = 0 \,..\,(\mathrm{IX}),$$

an equation of t dimensions in P; hence generally P will have t values, P being a linear function of x', y', z'; this equation will therefore represent generally t parallel planes, the direction-constants of all the planes being proportional to L, M, N.

Ex. 1. To find the equations to the asymptotic plane and asymptotic surface of the locus of the equation

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy + k = 0.$$

Here $\phi_n = \phi_2 = a\lambda^2 + b\mu^2 + c\nu^2 + 2a'\mu\nu + 2b'\nu\lambda + 2c'\lambda\mu$,

$$\phi_1 = 0,$$

and accordingly

$$D\phi_2 = 2\{x'(a\lambda + b'\nu + c'\mu) + y'(b\mu + c'\lambda + a'\nu) + z'(c\nu + a'\mu + b'\lambda)\}.$$

Hence the general equation to the asymptotic plane is

$$x'(a\lambda + b'\nu + c'\mu) + y'(b\mu + c'\lambda + a'\nu) + z'(c\nu + a'\mu + b'\lambda) = 0 \ldots (1),$$

λ, μ, ν, being subject to the condition

$$a\lambda^2 + b\mu^2 + c\nu^2 + 2a'\mu\nu + 2b'\nu\lambda + 2c'\lambda\mu = 0.\ldots(2).$$

Differentiating (1) and (2) with regard to the parameters λ, μ, ν, multiplying the latter of the resulting equations by an arbitrary quantity $\frac{1}{2}\rho$, adding and equating to zero the coefficients of $d\lambda, d\mu, d\nu$, we get

$$\begin{aligned} ax' + c'y' + b'z' + \rho(a\lambda + b'\nu + c'\mu) &= 0, \\ by' + a'z' + c'x' + \rho(b\mu + c'\lambda + a'\nu) &= 0, \\ cz' + b'x' + a'y' + \rho(c\nu + a'\mu + b'\lambda) &= 0. \end{aligned}$$

If we multiply the last three equations in order by λ, μ, ν, and then add them together, we shall get, by the aid of (2), an equation coinciding with (1): thus we see that these five equations are equivalent to only four independent ones. We may therefore confine our attention to the last three of them together with the equation (1). Multiplying the last three equations in order by x', y', z', and adding, we obtain, by virtue of (1),

$$ax'^2 + by'^2 + cz'^2 + 2c'x'y' + 2a'y'z' + 2b'z'x' = 0,$$

which is the equation to the asymptotic surface.

If the equation to the surface had been

$$ax^2 + by^2 + cz^2 + 2c'xy + 2a'yz + 2b'zx + 2a''x + 2b''y + 2c''z + k = 0,$$

it might be reduced to the form

$$a(x-f)^2 + b(y-g)^2 + c(z-h)^2 + 2c'(x-f)(y-g) \\ + 2a'(y-g)(z-h) + 2b'(z-h)(x-f) + k' = 0,$$

f, g, h, being the coordinates of its centre.

The equation to its asymptotic surface will then evidently be

$$a(x'-f)^2 + b(y'-g)^2 + c(z'-h)^2 + 2a'(y'-g)(z'-h) \\ + 2b'(z'-h)(x'-f) + 2c'(x'-f)(y'-g) = 0.$$

Ex. 2. To find the asymptotic planes and asymptotic developable surface of the surface

$$(x+a)(by^2 + cz^2) = mx^3 + n^4.$$

Here
$$\phi_n = \phi_3 = \lambda(b\mu^2 + c\nu^2) - m\lambda^3,$$
$$\phi_2 = a(b\mu^2 + c\nu^2),$$
$$D\phi_3 = x'(b\mu^2 + c\nu^2 - 3m\lambda^2) + y' \,.\, 2b\lambda\mu + z' \,.\, 2c\lambda\nu.$$

Hence, from the equation

$$D\phi_n + \phi_{n-1} = 0,$$

we have

$$(b\mu^2 + c\nu^2 - 3m\lambda^2)x' + 2b\lambda\mu y' + 2c\lambda\nu z' + a(b\mu^2 + c\nu^2) = 0 \ldots (1).$$

Also, from the equation $\phi_n = 0$, there is

$$\lambda(b\mu^2 + c\nu^2 - m\lambda^2) = 0,$$

which resolves itself into $\lambda = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2)$,

and $b\mu^2 + c\nu^2 - m\lambda^2 = 0 \ldots\ldots\ldots\ldots\ldots (3)$.

Taking the relation (2), we have, from (1),

$$(b\mu^2 + c\nu^2)x' + a(b\mu^2 + c\nu^2) = 0,$$

and therefore $$x' + a = 0,$$

the equation to an asymptotic plane. Thus in this example we have an instance of an asymptotic plane which does not form one of a family; the equation (2) therefore does not correspond to an asymptotic developable surface.

Again, combining the relation (3) with (1), we have

$$-2m\lambda x' + 2b\mu y' + 2c\nu z' + ma\lambda = 0 \ldots\ldots (4).$$

Differentiating (3) and (4) with regard to λ, μ, ν, and using an indeterminate multiplier ρ, we get

$$-\rho m\lambda - 2mx' + ma = 0,$$
$$\rho b\mu + 2by' = 0,$$
$$\rho c\nu + 2cz' = 0:$$

and from these equations

$$\lambda = \frac{a - 2x'}{\rho}, \quad \mu = -\frac{2y'}{\rho}, \quad \nu = -\frac{2z'}{\rho},$$

and therefore, from (3), we have for the asymptotic developable surface

$$4by'^2 + 4cz'^2 = m(a - 2x')^2.$$

Ex. 3. Take the surface

$$\frac{x^2}{a'^2}\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right)^3 + \frac{y^2}{b'^2}\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right)^2 + \frac{z^2}{c'^2}\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right)$$
$$+ \frac{x}{a''} + \frac{y}{b''} + \frac{z}{c''} + 1 = 0.$$

Here $$\phi_n = \phi_5 = \frac{\lambda^2}{a'^2}\left(\frac{\lambda}{a} + \frac{\mu}{b} + \frac{\nu}{c}\right)^3,$$

$$\phi_4 = \frac{\mu^2}{b'^2}\left(\frac{\lambda}{a} + \frac{\mu}{b} + \frac{\nu}{c}\right)^2,$$

$$\phi_3 = \frac{\nu^2}{c'^2}\left(\frac{\lambda}{a} + \frac{\mu}{b} + \frac{\nu}{c}\right),$$

$$\phi_2 = 0.$$

Putting $\phi_5 = 0$, we have

$$\frac{\lambda}{a} + \frac{\mu}{b} + \frac{\nu}{c} = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

and, by formula (IX), we obtain

$$\frac{\lambda^2}{a'^2}P^3 + \frac{\mu^2}{b'^2}P^2 + \frac{\nu^2}{c'^2}P = 0,$$

where $$P = \frac{x'}{a} + \frac{y'}{b} + \frac{z'}{c}.$$

Hence $$\frac{x'}{a} + \frac{y'}{b} + \frac{z'}{c} = 0$$

is the equation to an isolated asymptotic plane.

Taking the other factor, we have

$$\frac{\lambda^2}{a'^2} P^2 + \frac{\mu^2}{b'^2} P + \frac{\nu}{c'^2} = 0 \;\ldots\ldots\ldots\ldots(2),$$

which represents two parallel planes of a system, the parameters λ, μ, ν, being supposed to satisfy the equation (1).

Differentiating (1) and (2) with regard to λ, μ, ν, and using an arbitrary multiplier ρ, we get

$$\frac{\rho}{a} = \frac{\lambda P^2}{a'^2}, \quad \frac{\rho}{b} = \frac{\mu P}{b'^2}, \quad \frac{\rho}{c} = \frac{\nu}{c'^2},$$

and therefore, from (1) or (2),

$$\frac{a'^2}{a^2} + \frac{b'^2}{b^2} P + \frac{c'^2}{c^2} P^2 = 0,$$

the equation to the envelop of the planes (2), which consists therefore of two parallel planes.

Cambridge, December 31, 1846.

ON SOME THEOREMS OF USE IN THE INTEGRATION OF LINEAR DIFFERENTIAL EQUATIONS.

By the Rev. BRICE BRONWIN.

[*Extract from a letter to the Editor, received along with this Paper.*]

"Within the last few years, or since we became acquainted with the theorems

$$f(D+a)\,y = \varepsilon^{-ax} f(D)\,\varepsilon^{ax}\,y, \quad \text{and } f(D+\phi')\,y = \varepsilon^{-\phi} f(D)\,\varepsilon^{\phi}\,y,$$

(ϕ a function of n) the latter theorem given by Mr. Boole in the *Mathematical Journal*, the method of integrating Linear Differential Equations has undergone a great change.

"In the paper which I now send, I have given some very general theorems on this subject. Theorems of this kind are useful to transform an equation, or part of an equation; to enable us, when an equation is divisible by a factor, to operate with its reciprocal on the second member; and to exhibit general integral forms of equations."

Let D denote $\frac{d}{dx}$; and let ρ, π, and τ, be functions of x and D. Suppose these functions such that $\rho\pi\rho^{-1}u = \tau u$. Change u into $\rho\pi\rho^{-1}u$ in the first member, and into its equal τu in the second; and we have $\rho\pi^2\rho^{-1}u = \tau^2 u$. Repeat this operation, and we find successively $\rho\pi^3\rho^{-1}u = \tau^3 u$; $\rho\pi^4\rho^{-1} = \tau^4 u$; &c. We may find similar results for negative powers of π. The assumed equation gives $u = \tau^{-1}\rho\pi\rho^{-1}u$. Put $\rho\pi^{-1}\rho^{-1}u$ for u; there results $\rho\pi^{-1}\rho^{-1}u = \tau^{-1}u$. Change u into $\rho\pi^{-1}\rho^{-1}u$ in the first member of this, and into its equal $\tau^{-1}u$ in the second; and we have $\rho\pi^{-2}\rho^{-1}u = \tau^{-2}u$. Repeat the operation, and we find $\rho\pi^{-3}\rho^{-1}u = \tau^{-3}u$; $\rho\pi^{-4}\rho^{-1}u = \tau^{-4}u$; &c. Hence, $f(\pi)$ denoting a function of π which can be developed in integer powers of π, positive or negative,

If $\quad \rho\pi\rho^{-1}u = \tau u$; then also $\rho f(\pi)\,\rho^{-1}u = f(\tau)\,u. \ldots (a)$,

that is, the second of these is a necessary consequence of the first.

Let $\phi(D)^{-1}$ denote $\frac{1}{\phi(D)}$, X a function of x; and make $D = D' + D''$, D' operating upon y only, D'' upon x only.

By Taylor's theorem

$$\phi(D) = \phi(D' + D'') = \phi(D') + D''\phi'(D') + \tfrac{1}{2}D''^2\phi''(D') +.$$

Therefore

$$\phi(D)\,X\phi(D)^{-1}y = \{\phi(D') + D''\phi'(D') + \ldots\}\,X\phi(D)^{-1}y.$$

But $\quad \phi(D')\,X\phi(D)^{-1}y = X\phi(D)\,\phi(D)^{-1}y = Xy$;

$$D''\phi'(D')\,X\phi(D)^{-1}y = X'\phi'(D)\,\phi(D)^{-1}y\text{; \&c.}$$

Consequently, dropping the accent on D, as no longer necessary; we have

$$\phi(D)\,X\phi(D)^{-1}y = \left\{X + X'\frac{\phi'(D)}{\phi(D)} + \tfrac{1}{2}X''\frac{\phi''(D)}{\phi(D)} + \ldots\right\} y.$$

Now let X be an integer function of x of the degree n, $\phi(D)$ any function of D; or let X be any function of x, $\phi(D)$ an integer function of D of the degree n; the above series will terminate, and we shall have

$$\phi(D)\,X\phi(D)^{-1}y$$

$$= \left\{X + X'\frac{\phi'(D)}{\phi(D)} + \tfrac{1}{2}X''\frac{\phi''(D)}{\phi(D)} \ldots + \frac{X^{(n)}}{1.2\ldots n}\,\frac{\phi^{(n)}(D)}{\phi(D)}\right\} y.$$

Change in this y into $\phi(D)y$, and into $\psi(D)\,y$; we have also

$$\phi(D)\,Xy$$

$$=\left\{X\phi(D)+X'\phi'(D)+\tfrac{1}{2}X''\phi''(D)\ldots\ldots+\frac{X^{(n)}}{1.2\ldots n}\phi^{(n)}(D)\right\}y,$$

$$\phi(D)\,X\psi(D)\,\phi(D)^{-1}y=\phi(D)\,X\phi(D)^{-1}\psi(D)\,y$$

$$=\left\{X\psi(D)+X'\frac{\phi'(D)\,\psi(D)}{\phi(D)}\ldots\ldots+\frac{X^{(n)}}{1.2\ldots n}\,\frac{\phi^{(n)}(D)\,\psi(D)}{\phi(D)}\right\}y.$$

From the three last we have by (a) the three following:

$$f\left\{X+X'\frac{\phi'(D)}{\phi(D)}\ldots\ldots+\frac{X^{(n)}}{1.2\ldots n}\,\frac{\phi^{(n)}(D)}{\phi(D)}\right\}y$$

$$=\phi(D)\,f(X)\,\phi(D)^{-1}y\ldots\ldots(1),$$

$$f\left\{X\phi(D)+X'\phi'(D)\ldots\ldots+\frac{X^{(n)}}{1.2\ldots n}\phi^{(n)}(D)\right\}y$$

$$=f\{\phi(D)\,X\}\,y\ldots\ldots(2),$$

$$f\left\{X\psi(D)+X'\frac{\phi'(D)\,\psi(D)}{\phi(D)}\ldots\ldots+\frac{X^{(n)}}{1.2\ldots n}\,\frac{\phi^{(n)}(D)\,\psi(D)}{\phi(D)}\right\}y$$

$$=\phi(D)\,f\{X\psi(D)\}\,\phi(D)^{-1}y\ldots\ldots(3).$$

By the aid of such theorems as these we may sometimes put a differential equation under a more convenient form. And when the first member is divisible by a factor, they enable us to put the reciprocal of that factor under a suitable form for operating with it upon the second member. Thus if $\pi\tau u=U$, $\tau u=\pi^{-1}U$. Therefore if π^{-1} be the first member of any such equation as (1), (2), (3); by substituting the expression in the second member for it, we can operate upon U. They serve also to show us integrable forms.

Thus, U being supposed a function of x,

$$f\left\{X+X'\frac{\phi'(D)}{\phi(D)}\ldots\ldots+\frac{X^{(n)}}{1.2\ldots n}\,\frac{\phi^{(n)}(D)}{\phi(D)}\right\}y=U$$

is always integrable. For by (1) we have also

$$\phi(D)\,f(X)\,\phi(D)^{-1}y=U,\ \text{and}\ y=\phi(D)\,f(X)^{-1}\,\phi(D)^{-1}U,$$

which involves no operations but what can be performed.

$$f\left\{X\phi(D)+X'\phi'(D)\ldots\ldots+\frac{X^{(n)}}{1.2\ldots n}\phi^{(n)}(D)\right\}y=U$$

is often integrable; for by (2) we have $f\{\phi(D)\,X\}\,y=U$.

Make $\rho = \phi(D)X$; then $f(\rho)y = U$;

$$y = f(\rho)^{-1}U = \frac{U}{(\rho - a_1)(\rho - a_2)\ldots} = \frac{A_1}{\rho - a_1}U + \frac{A_2}{\rho - a_2}U + \ldots;$$

where a_1, a_2, &c. are the roots of $f(\rho) = 0$, and A_1, A_2, &c. are known quantities. But if

$$(\rho - a_1)y_1 = A_1 U;\ (\rho - a_2)y_2 = A_2 U;\ \text{\&c.};$$

then $$y_1 = \frac{A_1}{\rho - a_1}U;\quad y_2 = \frac{A_2}{\rho - a_2}U;\ \text{\&c.}$$

And $y = y_1 + y_2 + \ldots$. Therefore the integration is reduced to that of a number of others of the form

$$\phi(D)Xy_1 - a_1 y_1 = A_1 U;\quad \phi(D)Xy_2 - a_2 y_2 = A_2 U;\ \text{\&c.},$$

which are much more simple and easy to integrate.

$$f\left\{X\psi(D) + X'\frac{\phi'(D)\psi(D)}{\phi(D)}\ldots + \frac{X^{(n)}}{1.2\ldots n}\frac{\phi^{(n)}(D)\psi(D)}{\phi(D)}\right\}y = U$$

is under similar circumstances with the last. For by (3)

$$\phi(D)f\{X\psi(D)\}\phi(D)^{-1}y = U.$$

Make $$\phi(D)^{-1}U = U_1;\ \phi(D)^{-1}y = z;$$

and this becomes $f\{X\psi(D)\}z = U_1$; which, by making $\rho = X\psi(D)$, may be treated precisely as the last case, and we shall have to integrate

$$X\psi(D)z_1 - a_1 z_1 = A_1 U_1;\quad X\psi(D)z_2 - a_2 z_2 = A_2 U_1;\ \text{\&c.}$$

We now proceed to some more general theorems. Let A, A_1, &c., be functions of x; B, B_1, &c., functions of D; and make

$$\pi(x, D) = AB + A_1B_1 + A_2B_2 \ldots + A_mB_m,$$

so that the D in this function cannot operate upon the x contained in it. Now applying

$$\phi(D) = \phi(D') + D''\phi'(D') + \tfrac{1}{2}D''^2\phi''(D) + \ldots\ldots$$
$$\text{to } \pi(x, D)\,\phi(D)^{-1}y;$$

we find, as before, since D' operates upon y only, and D'' upon x only,

$$\phi(D)\,\pi(x, D)\,\phi(D)^{-1}y$$
$$= \left\{\pi(x, D) + \pi'(x, D)\frac{\phi'(D)}{\phi(D)} + \tfrac{1}{2}\pi''(x, D)\frac{\phi''(D)}{\phi(D)}\ldots\right.$$
$$\left. + \frac{\pi^{(n)}(x, D)}{1.2\ldots n}\frac{\phi^{(n)}(D)}{\phi(D)}\right\}y,$$

where π', π'', &c. denote the differential coefficients of the function taken relative to x. We have terminated the series at the term containing the n^{th} differential coefficients, because we shall make it to terminate there.

Let $\pi(x, D)$ be an integer function of x of the degree n, or an integer function of D of the degree n; that is, let A, A_1, &c. be integer functions of x of the degree n; B, B_1, &c. any functions of D; or let A, A_1, &c. be any functions of x, B, B_1, &c. integer functions of D of the degree n; and the preceding series will terminate as we have made it.

Change y into $\phi(D)y$, and then into $\psi(D)y$, in the last; and we have by (a), as before,

$$f\left\{\pi(x, D) + \pi'(x, D)\frac{\phi'(D)}{\phi(D)}\ldots\ldots + \frac{\pi^{(n)}(x, D)}{1.2\ldots n}\,\frac{\phi^{(n)}(D)}{\phi(D)}\right\} y$$

$$= \phi(D) f\pi(x, D)\,\phi(D)^{-1}y \ldots\ldots (4),$$

$$f\left\{\pi(x, D)\,\phi(D) + \pi'(x, D)\,\phi'(D) \ldots + \frac{\pi^{(n)}(x, D)}{1.2\ldots n}\,\phi^{(n)}(D)\right\} y$$

$$= f\{\phi(D)\,\pi(x, D)\}\, y \ldots\ldots (5),$$

$$f\left\{\pi(x, D)\,\psi(D) + \pi'(x, D)\,\frac{\phi'(D)\,\psi(D)}{\phi(D)}\ldots\ldots\right.$$

$$\left. + \frac{\pi^{(n)}(x, D)}{1.2\ldots n}\,\frac{\phi^{(n)}(D)\,\psi(D)}{\phi(D)}\right\} y$$

$$= \phi(D) f\{\pi(x, D)\,\psi(D)\}\,\phi(D)^{-1}y \ldots\ldots (6).$$

These may serve, in the same manner as (1), (2), and (3), either to put a given equation under another form; or when it is divisible by a factor, to enable us to operate with the reciprocal of that factor on the second member.

The three last theorems will likewise give us integrable equations; or equations, the integration of which will depend upon that of a number of others much more simple and easy to integrate.

$$f\left\{\pi(x, D) + \pi'(x, D)\,\frac{\phi'(D)}{\phi(D)}\ldots\ldots + \frac{\pi^{(n)}(x, D)}{1.2\ldots n}\,\frac{\phi^{(n)}(D)}{\phi(D)}\right\} y = U$$

will depend by (4) on

$$\phi(D) f\pi(x, D)\,\phi(D)^{-1}y = U.$$

Or, making

$$\phi(D)^{-1}U = U_1, \quad \phi(D)^{-1}y = z, \quad \text{on } f\pi(x, D)\,z = U_1.$$

Put $\rho = \pi(x, D)$; then $f(\rho)\,z = U_1$, which may be treated as already explained; and we shall have to integrate

$$(\rho - a_1)\,z_1 = A_1 U_1,\ \ (\rho - a_2)\,z_2 = A_2 U_1,\ \ \&\text{c.};$$

or, putting for ρ its value,

$$\{\pi(x, D) - a_1\}\,z_1 = A_1 U_1;\ \ \{\pi(x, D) - a_2\}\,z_2 = A_2 U;\ \ \&\text{c.}$$

$$f\left\{\pi(x, D)\,\phi(D) + \pi'(x, D)\,\phi'(D)\ldots\ldots + \frac{\pi^{(n)}(x, D)}{1.2\ldots n}\,\phi^{(n)}(D)\right\} y = U$$

$$f\left\{\pi(x, D)\,\psi(D) + \pi'(x, D)\,\frac{\phi'(D)\,\psi(D)}{\phi(D)}\ldots\ldots + \frac{\pi^{(n)}(x, D)}{1.2\ldots n}\,\frac{\phi^{(n)}(D)\,\psi(D)}{\phi(D)}\right\} y = U$$

depend, by (5) and (6), on

$$f\{\phi(D)\,\pi(x, D)\}\,y = U,$$

$$\phi(D)\,f\{\pi(x, D)\,\psi(D)\}\,\phi(D)^{-1}y = U.$$

And here, putting $\rho = \phi(D)\,\pi(x, D)$ in the first, and $\rho = \pi(x, D)\,\psi(D)$ in the second, and proceeding as heretofore; we shall have to integrate

$$\{\phi(D)\,\pi(x, D) - a_1\}\,y_1 = A_1 U,\ \&\text{c. in the first};$$

and $\{\pi(x, D)\,\psi(D) - a_1\}\,z_1 = A_1 U_1$, &c. in the second.

To give two very simple examples: in (5) make $f(x) = x$, $\phi(D) = D + a$, $\pi(x, D) = D + bx$, $U = 0$. We find from the first member

$$\frac{d^2y}{dx^2} + (a + bx)\frac{dy}{dx} + (1 + ax)\,by = 0.$$

And from the second $(D + a)(D + bx)\,y = 0$. Therefore

$$y = (D + bx)^{-1}(D + a)^{-1}\,0 = C(D + bx)^{-1}\,\varepsilon^{-ax}$$

$$= C\varepsilon^{-\frac{1}{2}bx^2}D^{-1}\,\varepsilon^{\frac{1}{2}bx^2-ax} = C\varepsilon^{-\frac{1}{2}bx^2}\!\int dx\,\varepsilon^{\frac{1}{2}bx^2-ax} + C'\,\varepsilon^{-\frac{1}{2}bx^2}.$$

The value of $(D + bx)^{-1}\,\varepsilon^{-ax}$ is known by a theorem given by Mr. Boole in this *Journal.*

Again, in (6) make

$$f(x) = x,\ \pi(x, D) = D + ax,\ \psi(D) = D^2 + bD,\ \phi(D) = D,\ U = 0.$$

Then the first member gives

$$\frac{d^3y}{dx^3} + (b + ax)\frac{d^2y}{dx^2} + (1 + bx)\,a\,\frac{dy}{dx} + aby = 0.$$

The second gives

$$D(D+ax)\,D(D+b)\,D^{-1}y=0, \text{ or } (D+ax)(D+b)\,y=C$$

by integrating and transposing D and $D+b$. Therefore

$$(D+b)y=(D+ax)^{-1}C=C\,\varepsilon^{-\frac{1}{2}ax^2}D^{-1}\,\varepsilon^{\frac{1}{2}ax^2}=C\,\varepsilon^{-\frac{1}{2}ax^2}\int dx\,\varepsilon^{\frac{1}{2}ax^2}+C'\,\varepsilon^{-\frac{1}{2}ax^2};$$

and $y=(D+b)^{-1}\left\{C\,\varepsilon^{-\frac{1}{2}ax^2}\int dx\,\varepsilon^{\frac{1}{2}ax^2}+C'\,\varepsilon^{-\frac{1}{2}ax^2}\right\}$

$$=C\,\varepsilon^{-bx}\int dx\,\varepsilon^{bx-\frac{1}{2}ax^2}\int dx\,\varepsilon^{\frac{1}{2}ax^2}+C'\,\varepsilon^{-bx}\int dx\,\varepsilon^{bx-\frac{1}{2}ax^2}+C''\varepsilon^{-bx}.$$

We now proceed to some different formulæ.

Since $\pi(D)=\pi(D')+D''\pi'(D')+\frac{1}{2}D''^2\pi''(D')+\dots\,;$ we find, by operating with each member on $\phi(x)\,y$,

$$\pi(D)\,\phi(x)\,y=\{\phi(x)\,\pi(D)+\phi'(x)\,\pi'(D)+\tfrac{1}{2}\phi''(x)\,\pi''(D)\dots\}\,y,$$

dropping the accent, as not being further wanted.

Suppose now $\pi(D)$ an integer function of D of the degree n, or $\phi(x)$ an integer function of x of the same degree; and the last becomes

$\pi(D)\,\phi(x)\,y$

$$=\left\{\phi(x)\,\pi(D)+\phi'(x)\,\pi'(D)\dots\dots+\frac{\phi^{(n)}(x)}{1.2\dots n}\,\pi^{(n)}(D)\right\}y.$$

Whence also $\phi(x)^{-1}\,\pi(D)\,\phi(x)\,y$

$$=\left\{\pi(D)+\frac{\phi'(x)}{\phi(x)}\,\pi'(D)\dots\dots+\frac{\phi^{(n)}(x)}{1.2\dots n.\phi(x)}\,\pi^{(n)}(D)\right\}y,$$

$\phi(x)^{-1}\,\psi(x)\,\pi(D)\,\phi(x)\,y$

$$=\left\{\psi(x)\,\pi(D)+\frac{\psi(x)\,\phi'(x)}{\phi(x)}\,\pi'(D)\dots+\frac{\psi(x)\,\phi^{(n)}(x)}{1.2\dots n.\phi(x)}\,\pi^{(n)}(D)\right\}y.$$

From these we obtain by (a)

$$f\left\{\phi(x)\,\pi(D)+\phi'(x)\,\pi'(D)\dots\dots+\frac{\phi^{(n)}(x)}{1.2\dots n}\,\pi^{(n)}(D)\right\}y$$

$$=f\{\pi(D)\,\phi(x)\}\,y\dots\dots(7),$$

$$f\left\{\pi(D)+\frac{\phi'(x)}{\phi(x)}\,\pi'(D)\dots\dots+\frac{\phi^{(n)}(x)}{1.2\dots n.\phi(x)}\,\pi^{(n)}(D)\right\}y$$

$$=\phi(x)^{-1}f\pi(D)\,\phi(x)\,y\dots\dots(8),$$

$$f\left\{\psi(x)\,\pi(D)+\frac{\psi(x)\,\phi'(x)}{\phi(x)}\,\pi'(D)\dots+\frac{\psi(x)\,\phi^{(n)}(x)}{1.2\dots n.\,\phi(x)}\,\pi^{(n)}(D)\right\}y$$

$$=\phi(x)^{-1}f\{\psi(x)\,\pi(D)\}\,\phi(x)\,y\dots\dots(9).$$

The second of these gives an integrable equation. For

$$f\left\{\pi(D)+\frac{\phi'(x)}{\phi(x)}\pi'(D)\ldots.+\frac{\phi^{(n)}(x)}{1.2\ldots n.\phi(x)}\pi^{(n)}(D)\right\}y=U$$

gives $\phi(x)^{-1}f\pi(D)\,\phi(x)\,y=U$, and $y=\phi(x)^{-1}f\{\pi(D)\}^{-1}\phi(x)U$, where all the operations can be performed.

The first members of (7) and (9) being equal to U, the second members may be treated as before by making $\rho=\pi(D)\,\phi(x)$ in the first, and $\rho=\psi(x)\,\pi(D)$ in the last; and changing in the last $\phi(x)U$ into U_1, $\phi(x)y$ into z; when the integration will be reduced to that of others much more simple.

We may also employ the three last theorems to transform an equation. And when the first member of an equation is divisible by a factor of the form of the first member of either of these, the second member of these will put it under a form for operating with it by actual division; however, with (7) and (9), this may not always be practicable.

Now we make $\pi(x,D)=A+A_1D+A_2D^2\ldots.+A_nD^n$; A, A_1, &c. being functions of x. Operating with this on $\phi(x)\,y$, we easily find

$$\pi(x,D)\,\phi(x)\,y=\{B+B_1D+B_2D^2\ldots.+B_nD^n\}\,y;$$

where $B=A\phi(x)+A_1\phi'(x)+A_2\phi''(x)\ldots.+A_n\phi^{(n)}(x)$;

$$B_1=A_1\phi(x)+2A_2\phi'(x)\ldots.+nA_n\phi^{(n-1)}(x);$$

$$B_2=A_2\phi(x)+3A_3\phi'(x)+6A_4\phi''(x)\ldots.\text{ to }B_n=A_n\phi(x).$$

Make $\quad C=B\,\phi(x)^{-1},\ C_1=B_1\,\phi(x)^{-1}$ &c.;

and $G\quad =B\,\psi(x)\phi(x)^{-1},\ G_1=B_1\psi(x)\,\phi(x)^{-1}$, &c.,

and we have

$$\phi(x)^{-1}\pi(x,D)\phi(x)y=\{C+C_1D\ldots.+C_nD^n\}y$$

$$\phi(x)^{-1}\psi(x)\pi(x,D)\phi(x)y=\{G+G_1D\ldots.+G_nD^n\}y.$$

From the above we have by (a)

$$f\{B+B_1D\ldots.+B_nD^n\}y=f\{\pi(x,D)\phi(x)\}y\ \ldots\ldots\ (10),$$

$$f\{C+C_1D\ldots.+C_nD^n\}y=\phi(x)^{-1}f\pi(x,D)\phi(x)y\ \ldots\ (11),$$

$$f\{G+G_1D\ldots.+G_nD^n\}y=\phi(x)^{-1}f\{\psi(x)\pi(x,D)\}\,\phi(x)y.(12).$$

These in their more simple forms may sometimes enable us to put an equation under a better form for integration; but they are particularly useful, when an equation is divisible by a factor, to enable us to operate with its

reciprocal on the second member: and they are the best for that purpose that can be found.

It is obvious that the three last may be treated as the nine preceding ones have been when we make the first member equal to U; but in each of them we shall have to employ the symbol ρ, except when $f(x) = x$ is the form of the function f. But as the mode of proceeding has been already sufficiently explained, it is unnecessary to repeat it.

The exponential form $\varepsilon^{\phi(D)}$, $\varepsilon^{\phi(x)}$ is of course included in the general form $\phi(D)$, $\phi(x)$. And in the last six theorems we may change $\phi(x)$ into $\varepsilon^{\phi(x)}$; but it is not convenient, in the very general form given to these theorems, to employ in them this form of the function. Of the form $\varepsilon^{\phi(D)}$ no use can be made, except to exhibit the general form of the first member of the equation by expanding the second. Thus in (1) make

$$\rho = X + X' \frac{\phi'(D)}{\phi(D)} + \ldots\ldots;$$

it becomes $$f(\rho)y = \phi(D)f(X)\phi(D)^{-1}y;$$

change $\phi(D)$ into $\varepsilon^{\phi(D)}$, and suppose this change made in ρ, we have $$f(\rho)y = \varepsilon^{\phi(D)}f(X)\,\varepsilon^{-\phi(D)}y.$$

But $\varepsilon^{\phi(D)} = \varepsilon^{\phi(D')+D''\phi'(D)+\ldots} = \varepsilon^{\phi(D')+B} = \varepsilon^{\phi(D')}\,\varepsilon^{B}$ suppose.

Substitute this value. Then as $\varepsilon^{\phi(D')}$ destroys $\varepsilon^{-\phi(D)}$, we shall have $$f(\rho)y = \varepsilon^{B}f(X)y.$$

If we can resolve and integrate $f(\rho)y = U$; then $\varepsilon^{B}f(X)y = U$, or $\{1 + B + \frac{1}{2}B^2\ldots\}f(X)y = U$, by actually operating with B, B^2, &c. on $f(X)y$ will give a general form of the equation which can be so integrated. But this requires that X should be an integral function of x, or the equation would have an infinite number of terms.

It is my wish in another paper to treat of such forms as

$$\phi(x, D)^{-1}\pi(x, D)\phi(x, D)y,$$

and shall conclude this by observing that the integral of every equation $\tau y = U$ furnishes a theorem for the transformation of an operating factor.

Gunthwaite Hall, March 16, 1847.

ON THE CONDITION THAT A PLANE SHOULD TOUCH A SURFACE ALONG A CURVE LINE.

By the Rev. George Salmon.

[*To the Editor of the Cambridge and Dublin Mathematical Journal.*]

My attention having been lately directed (by a reference in Gregory's *Solid Geometry*, p. 214) to an article with the above title in the *Cambridge Mathematical Journal* (vol. II. p. 22), I venture with some diffidence to raise a question as to the *sufficiency* of the condition there found.

To save your readers the trouble of referring to another volume, I give here the principal steps of the investigation alluded to.

The equation of any tangent plane is

$$Ux' + Vy' + Wz' = P,$$

(where U, V, W, are the differential coefficients with regard to x, y, z of the equation of the surface, and $P = Ux + Vy + Wz$). This tangent plane will be constant if $\frac{U}{P}$, $\frac{V}{P}$, $\frac{W}{P}$ are constant. The differentials of each of these quantities put $= 0$ give the three conditions

$$\frac{dU}{U} = \frac{dV}{V} = \frac{dW}{W} = \frac{dP}{P},$$

and the first two of these are shewn to be equivalent to the condition

$$U^2(vw - u'^2) + V^2(wu - v'^2) + W^2(uv - w'^2) \\ + 2VW(v'w' - uu') + 2WU(w'u' - vv') \\ + 2UV(u'v' - ww') = 0 \ldots\ldots (1),$$

$$u \text{ being} = \frac{du}{dx}, \quad u' = \frac{dv}{dz} = \frac{dw}{dy}, \text{ \&c.},$$

or if z be given explicitly as a function of x and y to

$$rt - s^2 = 0.$$

Now if we admit this condition to be not only *necessary* but *sufficient*, it will follow that *every* surface contains a series of points whose tangent planes will touch the surface along plane curves, the points being found by combining equation (1) with the equation of the surface.

Again, it is known that $rt - s^2 = 0$ is the condition that the two directions which give an infinite radius of curvature at any point for which it is satisfied, should coincide (Leroy,

Art. 391), or that the point should be what Gergonne, after Dupin, has called a *parabolic* point on the surface (*Annales de Mathématique*, vol. XXI. p. 233). Indeed equation (1) is identical with that given by M. Gergonne, in the memoir cited, as the locus of the parabolic points on the surface. If therefore we admit that this is the only condition necessary that a plane should touch along a curve line, it appears to follow that the tangent plane at every parabolic point must touch along a curve line, and indeed that the curve represented by equation (1) and the equation of the surface must break up into a number of plane curves.

I believe however that the article in the *Cambridge Mathematical Journal* would have been more accurately headed "On the condition that a plane should touch a surface in *two consecutive* points." This is all which seems to me to follow from the investigation I have quoted, and I proceed to attempt to shew by a different method that this is all that is true for parabolic points.

The origin being taken on any surface, its equation is

$$u_1 + u_2 + u_3 + \&c. = 0,$$

(where u_1 is used to denote the terms of the first degree, &c.); then $u_1 = 0$ is the equation of the tangent plane, and the origin will be a parabolic point if when we combine this equation with $u_2 = 0$ the result is a perfect square. Choosing then for axes, the normal, the direction of evanescent curvature, and a perpendicular to them, it will suffice to write the equation of the surface

$$z + x^2 + u_3 + \&c. = 0.$$

Now if we transfer the origin to any consecutive point, we obtain

$$\text{the new } u_1 = z + 2xx'$$

$$\text{the new } u_2 = x^2 + \frac{du_3}{dx}\,x' + \frac{du_3}{dy}\,y'.$$

Hence the tangent at any consecutive point will pass through the axis of y; and if we take the consecutive point on this axis, the new tangent plane will coincide with the old one. But this consecutive point will not be in general a point on the parabolic curve, because the quantity

$$x^2 + \frac{du_3}{dx}\,x_1 + \frac{du_3}{dy}\,y' = 0$$

is not in general a perfect square. In fact the direction of the consecutive point on the parabolic curve is determined

from the equation $\frac{du_3}{dy^2} = 0$, which will not in general coincide with the direction $x = 0$.

Hence I conclude that a parabolic point is such that the tangent plane remains constant in passing to a consecutive point *in a certain direction*, but that it is not in general true that two consecutive points *on the parabolic curve* have the same tangent plane.*

ON THE NUMBER OF NORMALS WHICH CAN BE DRAWN FROM A GIVEN POINT TO A GIVEN SURFACE.

By the Rev. George Salmon.

On turning to the Second Volume of the *Cambridge Mathematical Journal* in search of the article just alluded to, I noticed (p. 13) a paper with the above title, translated from an article by M. Terquem in *Liouville's Journal*. It is sometimes useful to apply both geometrical and analytical methods to the same problem, each throwing light on the results of the other; I therefore send a solution of the same question which I had found some time ago, and which serves to shew when exceptions to M. Terquem's results will occur.

The number of normals which can be drawn to a surface from a point must be constant whatever be the point; it suffices therefore to consider the case where the point is at an infinite distance. Now the number of normals parallel to a given line is equal to the number of tangent planes which can be drawn parallel to a given plane or to the degree of the reciprocal surface. But this is not the total number of normals which can be drawn through a point at infinity, for we shall shew that normals may pass through it themselves lying altogether at an infinite distance. Take any point of the surface at infinity, then the normal at this point will be at an infinite distance, and will of course be a normal in the plane section at infinity. Hence, as many such normals will pass through the given point as there can be drawn normals

* [There will always be a real "parabolic curve" when the surface is such that over a finite portion of it there is hyperbolic contact, and over another portion, also finite, there is elliptic contact, with the tangent plane. It is not generally true that the curve separating the two portions, or the curve of parabolic contact, is plane. It is easy to obtain examples illustrative of this, by considering a curve as generating, by its motion, a surface. The loci of its points of inflection, and of those points where the motion of the generating curve is in its osculating plane, are the "parabolic curves" of the surface.]

from a given point in a plane section to that curve. The total number of normals is therefore equal to the degree of the reciprocal surface + the number of normals which can be drawn to a plane section.

The question of surfaces is thus reduced to that of curves. But the number of normals which can be drawn to a curve from a point at infinity is in like manner equal to the degree of the reciprocal curve + the number of normals to the curve which lie at infinity, or + the number of points in which the curve is cut by a straight line at infinity, or + the degree of the curve. Hence the number of normals which can be drawn from a point to a curve is equal to *the sum of the degrees of the curve and its reciprocal.* And the number of normals which can be drawn to a surface is equal to *the sum of the degrees of the surface and its reciprocal + the degree of the reciprocal of a plane section.* This coincides with M. Terquem's results (m^2 and $m^3 - m^2 + m$) if the curve or surface have no multiple points.

And in order to be sure that this number meets no further reduction, it is obviously necessary to attend to the nature of the section of the curve or surface made by a line or plane at infinity. Thus, for example, if the curve be of a parabolic nature, having the line at infinity for a tangent, the number of tangents which can be drawn parallel to a given line is diminished by one for every point at which the line at infinity touches the curve. Thus, though we can in general draw four normals to a conic section, we can only draw three to a parabola.

The relation of tangent and normal being lost in projection, I have sometimes found convenient to substitute for it another more general one.

Given a line and two fixed points A, B, on it; if the tangent at any point P of the curve meet this line in the point T, and we take N a fourth harmonic to A, B, T, the line PN will be analogous to the normal and will coincide with it if the line AB be at infinity and the points A, B, the imaginary points in which any circle meets it.

Or more simply thus: Given any conic section; if we take the pole with regard to it of any tangent to a curve and join this pole to the point of contact, the joining line will be analogous to the normal and will coincide with it if the conic section be a circle of infinite radius.

Any projective properties of normals will be true for these analogous lines.

Similar generalizations can be made for surfaces.

Trinity College, Dublin, November 9, 1847.

DEMONSTRATION OF A GEOMETRICAL THEOREM OF JACOBI'S.

By Arthur Cayley.

The theorem in question (*Crelle*, tom. xii. p. 137) may be thus stated:

"If a cone be circumscribed about a surface of the second order, the focal lines of the cone are generating lines of a surface of the second order confocal to the given surface and which passes through the vertex of the cone."

Let (α, β, γ) be the coordinates of the given point,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

the equation to the given surface. The equation of the circumscribed cone referred to its vertex is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1\right) - \left(\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2}\right)^2 = 0,$$

whence it is easily seen that the equation of the supplementary cone (*i.e.* the cone generated by lines through the vertex at right angles to the tangent planes of the cone in question) is

$$(\alpha x + \beta y + \gamma z)^2 - a^2x^2 - b^2y^2 - c^2z^2 = 0.*$$

Suppose we have identically

$$(\alpha x + \beta y + \gamma z)^2 - a^2x^2 - b^2y^2 - c^2z^2 - h\,(x^2 + y^2 + z^2)$$
$$= (lx + my + nz)\,(l'x + m'y + n'z).$$

$lx + my + nz = 0$ will determine the direction of one of the cyclic planes of the supplementary cone, and hence taking the centre for the origin the equations of the focal lines of the circumscribed cone are

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}.$$

It only remains therefore to determine the values of l, m, n from the last equation but one. The condition which expresses that the first side of this equation divides itself into factors is easily reduced to

$$\frac{\alpha^2}{a^2 + h} + \frac{\beta^2}{b^2 + h} + \frac{\gamma^2}{c^2 + h} = 1 \ldots\ldots\ldots\ldots (A).$$

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* $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gxz + 2Hxy = 0$ being the equation of the first cone, that of the supplementary cone is $\mathfrak{A}x^2 + \mathfrak{B}y^2 + \mathfrak{C}z^2 + 2\mathfrak{F}yz + 2\mathfrak{G}xz + 2\mathfrak{H}xy = 0$, these letters having the same signification as in *Mathematical Journal*, tom. ii, p. 274.

Next, since the equation is identical, write

$$x = \frac{\alpha}{a^2 + h}, \quad y = \frac{\beta}{b^2 + h}, \quad z = \frac{\gamma}{c^2 + h},$$

we deduce
$$0 = \frac{l\alpha}{a^2 + h} + \frac{m\beta}{b^2 + h} + \frac{n\gamma}{c^2 + h} \quad \ldots\ldots\ldots \text{(B)}.$$

Again, putting

$$x = \frac{l}{a^2 + h}, \quad y = \frac{m}{b^2 + h}, \quad z = \frac{n}{c^2 + h},$$

the whole equation divides by

$$\left(\frac{ll'}{a^2 + h} + \frac{mm'}{b^2 + h} + \frac{nn'}{c^2 + h} + 1\right),$$

a factor whose value is easily seen to be (-1). And rejecting this, we have

$$\frac{l^2}{a^2 + h} + \frac{m^2}{b^2 + h} + \frac{n^2}{c^2 + h} = 0 \quad \ldots\ldots\ldots \text{(C)}.$$

Thus of the three equations (A), (B), and (C), the first determines h, and the remaining two give the ratios $l:m:n$. It is obvious that

$$\frac{x^2}{a^2 + h} + \frac{y^2}{b^2 + h} + \frac{z^2}{c^2 + h} = 1$$

is the equation of the surface confocal with the given surface which passes through the point (α, β, γ). The generating lines at this point are found by combining this equation with that of the tangent plane at the same point, viz.

$$\frac{\alpha x}{a^2 + h} + \frac{\beta y}{b^2 + h} + \frac{\gamma z}{c^2 + h} = 1.$$

And since these two equations are satisfied by $x = \alpha + lr$, $y = \beta + mr$, $z = \gamma + nr$, if l, m, n are determined by the equations above, it follows that the focal lines of the cone are the generating lines of the surface, the theorem which was to be demonstrated. It is needless to remark that of the three confocal surfaces, the hyperboloid of one sheet has alone real generating lines; this is as it should be, since a cone has six focal lines, of which four are always impossible.

58, *Chancery Lane, Feb.* 16, 1847.

ON THE THEORY OF ELLIPTIC FUNCTIONS.

By Arthur Cayley.

[*Continued from* Vol. II. p. 266.]

We have seen that the equation

$$n.(n-1)\,x^2z + (n-1)(ax-2x^3)\frac{dz}{dx} + (1-ax^2+x^4)\frac{d^2z}{dx^2}$$

$$-\,2n.(a^2-4)\frac{dz}{da} = 0$$

is integrable in the case of n an odd number, in the form $z = B_0 + B_1x^2 \,..+ B_{\frac{1}{2}(n-1)}x^{n-1}$; and the coefficients at the beginning of the series have already been determined; to find those at the end of it, the most convenient mode of writing the series will be

$$z = \mu\Sigma\,\frac{(-)^r.D_r}{1.2..(2r+1)}\,x^{n-1-2r}.$$

And then the coefficients D_r are determined by

$$D_{r+2} = (2r+3)(n-2r-3)\,D_{r+1}a - 2n(a^2-4)\frac{dD_{r+1}}{da}$$

$$-\,(2r+3)(2r+2)(n-2r-2)(n-2r-1)\,D_r.$$

And the first coefficients are

$$D_0 = 1,$$

$$D_1 = (n-1)\,a,$$

$$D_2 = 2(n-1)(n+6) + (n-1)(n-9)\,a^2,$$

$$D_3 = 6(n-1)(n-9)(n+10)\,a + (n-1)(n-9)(n-25)\,a^3,$$

$$\begin{aligned}D_4 = &-36(n-1)(n^3-13n^2+36n+420)\\ &+12(n-1)(n-9)(n-25)(n+14)\,a^2\\ &+(n-1)(n-9)(n-25)(n-49)\,a^4,\end{aligned}$$

$$\begin{aligned}D_5 = &-12(n-1)(n-9)(47n^3-355n^2+3188n+31500)\,a\\ &+20(n-1)(n-9)(n-25)(n-49)(n+18)\,a^3\\ &+(n-1)(n-9)(n-25)(n-49)(n-81)\,a^5,\end{aligned}$$

$$\begin{aligned}D_6 = &-24(n-1)(n-9)(23n^4+2375n^3-14638n^2\\ &\qquad\qquad +116100n+693000)\\ &-12(n-1)(n-9)(493n^4-8882n^3+70317n^2\\ &\qquad\qquad -361641n-7276500)\,a^2\\ &+30(n-1)(n-9)(n-25)(n-49)(n-81)(n+22)\,a^4\\ &+(n-1)(n-9)(n-25)(n-36)(n-49)(n-81)(n-121)\,a^6,\end{aligned}$$

&c.

And, in general,

$$D_r = (n-1)(n-9)\ldots\{n-(2r-1)^2\}\,a^r$$
$$+\,r(r-1)(n-1)(n-9)\ldots\{n-(2r-3)^2\}$$
$$(n+4r-2)\,a^{r-2},$$
&c.

(where however the next term does not contain the factor $(n-1)(n-9)\ldots\{n-(2r-5)^2\}$).

In the case when $n=\nu^2$, then in order that the constant term may reduce itself to unity, we must assume

$$\mu = (-)^{\frac{\nu-1}{2}}\,\nu;$$

this is evident from what has preceded.

58, *Chancery Lane, Nov.* 5, 1847.

NOTES ON THE ABELIAN INTEGRALS.—JACOBI'S SYSTEM OF DIFFERENTIAL EQUATIONS.

By Arthur Cayley.

The theory of elliptic functions depends, it is well known, on the differential equation $\dfrac{dx}{\sqrt{(fx)}}+\dfrac{dy}{\sqrt{(fy)}}=0$, ($fx$ denoting a rational and integral function of the fourt horder), the integral of which was discovered by Euler, though first regularly derived from the differential equation by Lagrange. The theory of the Abelian integrals depends in like manner, as is proved by Jacobi, in the memoir "Considerationes generales de transcendentibus Abelianis" (*Crelle*, tom. IX. p. 394) to depend, upon the system of equations

$$\Sigma\,\frac{dx}{\sqrt{(fx)}}=0,\quad \Sigma\,\frac{xdx}{\sqrt{(fx)}}=0,\ldots\Sigma\,\frac{x^{n-2}dx}{\sqrt{(fx)}}=0\ldots(1),$$

where fx is a rational and integral function of the order $2n-1$ or $2n$, and the sums Σ contain n terms.

The integration of this system of equations is of course virtually comprehended in Abel's theorem; the problem was to obtain $(n-1)$ integrals each of them containing a single independent arbitrary constant. One such integral was first obtained by Richelot (*Crelle*, tom. XXIII. p. 354), "Ueber die Integration eines merkwürdigen Systems Differentialgleichungen," by a method founded on that of Lagrange for the solution of Euler's equation; and a second integral very

ingeniously deduced from it. A complete system of integrals in the required form is afterwards obtained, not by direct integration, but by means of Abel's theorem: there is this objection to them, however, that any one of them contains two roots of the equation $fx = 0$. The next paper on the subject is one by Jacobi, "Demonstratio Nova theorematis Abeliani" (*Crelle*, tom. XXIV. p. 28), in which a complete system of equations is deduced by direct integration, each of which contains only a single root of the equation $fx = 0$. But in Richelot's second memoir "Einige neue Integralgleichungen des Jacobischen Systems Differentialgleichungen" (*Crelle*, tom. XXV. p. 97), the equations are obtained by direct integration in a form not involving any of the roots of this equation; the method employed in obtaining them being in a great measure founded upon the memoir just quoted of Jacobi's. The following is the process of integration.

Denoting the variables by $x_1, x_2 \ldots . x_n$, and writing

$$Fa = (a - x_1)(a - x_2) \ldots . (a - x_n),$$

so that

$$F'x_1 = (x_1 - x_2) \ldots . (x_1 - x_n),$$

&c.

Then the system of differential equations is satisfied by assuming that $x_1, x_2 \ldots . x_n$ are functions of a new variable t, determined by the equations

$$\frac{dx_1}{dt} = \frac{\sqrt{(fx_1)}}{F'x_1} \text{ \&c.}$$

(In fact these equations give $\Sigma \dfrac{dx}{\sqrt{(fx)}} = dt \;\; \Sigma \dfrac{1}{F'x} = 0$, &c.)

From these we deduce, by differentiation,

$$\frac{d^2x_1}{dt^2} = \tfrac{1}{2} \frac{d}{dx_1} \frac{fx_1}{(F'x_1)^2} + \frac{\sqrt{(fx_1)}}{F'x_1} \Sigma' \frac{\sqrt{(fx)}}{(x_1 - x) F'x}$$

(where Σ' refers to all the roots except x_1) and a set of analogous equations for $x_2, x_3 \ldots . x_n \ldots \ldots$

Dividing this by $a - x_1$, where a is arbitrary, and reducing by

$$\frac{1}{(a - x_1)(x_1 - x)} = \frac{1}{2(a - x)(a - x_1)} \left(1 - \frac{x + x_1 - 2a}{x_1 - x}\right),$$

we have

$$\frac{1}{a - x_1} \frac{d^2x_1}{dt^2} = \frac{1}{2(a - x_1)} \frac{d}{dx_1} \frac{fx_1}{(x' F'_1)^2}$$

$$+ \tfrac{1}{2} \frac{\sqrt{(fx_1)}}{(a - x_1) F'x_1} \Sigma' \frac{\sqrt{(fx)}}{(a - x) F'x} - \tfrac{1}{2} \Sigma' \frac{\sqrt{(fx)}\sqrt{(fx_1)}}{F'x \, F'x_1} \frac{(x_1 + x - 2a)}{(a - x)(a - x_1)(x_1 - x)}$$

i.e. $$\frac{1}{a-x_1}\frac{d^2x_1}{dt^2} = \frac{1}{2(a-x_1)}\frac{d}{dx_1}\frac{fx_1}{(F'x_1)^2}$$

$$+\tfrac{1}{2}\frac{\surd(fx_1)}{(a-x_1)F'x_1}\Sigma\frac{\surd(fx)}{(a-x)F'x} - \tfrac{1}{2}\frac{fx_1}{(a-x_1)^2(F'x_1)^2}$$

$$-\tfrac{1}{2}\Sigma'\frac{\surd(fx)\surd(fx_1)}{F'x\,F'x_1}\frac{(x_1+x-2a)}{(a-x)(a-x_1)(x_1-x)}$$

And taking the sum of all the equations of this form, the last term disappears on account of the factor $x_1 - x$ in the denominator, and the result is

$$\Sigma\frac{1}{a-x}\frac{d^2x}{dt^2} = \tfrac{1}{2}\Sigma\frac{1}{a-x}\frac{d}{dx}\frac{fx}{(F'x)^2}$$

$$+\tfrac{1}{2}\left\{\Sigma\frac{\surd(fx)}{(a-x)F'x}\right\}^2 - \tfrac{1}{2}\Sigma\frac{fx}{(a-x)^2(F'x)^2}.$$

This being premised, assume

$$y = \surd(Fa),$$

which, by differentiation, gives

$$\frac{dy}{dt} = -\tfrac{1}{2}y\Sigma\frac{1}{a-x}\frac{dx}{dt} = -\tfrac{1}{2}y\Sigma\frac{\surd(fx)}{(a-x)F'x},$$

and thence

$$\frac{d^2y}{dt^2} = -\tfrac{1}{2}\frac{dy}{dt}\Sigma\frac{1}{a-x}\frac{dx}{dt} + \tfrac{1}{2}y\Sigma\frac{1}{(a-x)^2}\left(\frac{dx}{dt}\right)^2 - \tfrac{1}{2}y\Sigma\frac{1}{(a-x)}\frac{d^2x}{dt^2},$$

i.e.

$$\frac{d^2y}{dt^2} = \tfrac{1}{4}y\left(\Sigma\frac{\surd(fx)}{(a-x)F'x}\right)^2 - \tfrac{1}{2}y\Sigma\frac{fx}{(a-x)^2(F'x)^2} - \tfrac{1}{2}y\Sigma\frac{1}{a-x}\frac{d^2x}{dt^2}.$$

Or substituting the preceding value of $\Sigma\dfrac{1}{a-x}\dfrac{d^2x}{dt^2}$,

$$\frac{d^2y}{dt^2} = -\tfrac{1}{4}y\Sigma\frac{fx}{(a-x)^2(F'x)^2} - \tfrac{1}{4}y\Sigma\frac{1}{a-x}\frac{d}{dx}\frac{fx}{(F'x)^2};$$

i.e. $$4\frac{d^2y}{dt^2} + y\left\{\Sigma\frac{fx}{(a-x)^2(F'x)^2} + \Sigma\frac{1}{(a-x)}\frac{d}{dx}\frac{fx}{(F'x)^2}\right\} = 0.$$

Now the fractional part of $\dfrac{fa}{(Fa)^2}$ is equal to

$$\Sigma\frac{fx}{(a-x)^2(F'x)^2} + \Sigma\frac{1}{a-x}\frac{d}{dx}\frac{fx}{(F'x)^2}.$$

Also if L be the coefficient of x^{2n} in $f\alpha$, the integral part is simply equal to L, (since $(F\alpha)^2$ is a function of the order $2n$, in which the coefficient of α^{2n} is unity). Hence the coefficient of y in the last equation is simply

$$\frac{f\alpha}{(F\alpha)^2} - L, = \frac{f\alpha}{y^4} - L.$$

Or we have $$4\frac{d^2y}{dt^2} + y\left(\frac{f\alpha}{y^4} - L\right) = 0,$$

i. e. multiplying by the factor $2\frac{dy}{dt}$, and integrating

$$4\left(\frac{dy}{dt}\right)^2 - \frac{f\alpha}{y^2} - Ly^2 = C.$$

Or replacing y and $\frac{dy}{dt}$ by their values

$$\sqrt{(F\alpha)} \text{ and } - \tfrac{1}{2}\sqrt{(F\alpha)}\,\Sigma\,\frac{\sqrt{(fx)}}{(\alpha - x)\,F'x},$$

we have $$F\alpha\left\{\Sigma\,\frac{\sqrt{(fx)}}{(\alpha - x)\,F'x}\right\}^2 - \frac{f\alpha}{F\alpha} - LF\alpha = C,$$

for one of the integrals of the proposed system of equations, and since α is arbitrary, the complete system is obtained by giving any $(n - 1)$ particular values to α, and changing the value of the constant of integration C: or by expanding the first side of the equation in terms of α, and equating the different coefficients to arbitrary constants. The *à posteriori* demonstration that all the results so obtained are equivalent to $(n - 1)$ independent equations would probably be of considerable interest.

NOTE SUR LA THÉORIE DES FONCTIONS ELLIPTIQUES.

Par *M.* C. Hermite.

J'ai lu avec le plus vif interet, le beau travail de Mr. Cayley, qui a réussi à faire découler toute la théorie des fonctions elliptiques, de la considération délicate des produits infinis doubles J'avais découvert de mon côté le point de vue suivant, plus voisin peut être encore, de l'idée fondamentale de la double périodicité dans les fonctions analytiques.

En désignant par

$$\sum_{-\infty}^{+\infty}{}_m a_m e^{2m\frac{i\pi x}{a}}, \quad \text{et} \quad \sum_{-\infty}^{+\infty}{}_m b_m e^{2m\frac{i\pi x}{a}},$$

les expressions générales de deux fonctions périodiques simples, dont la période est a, assujéties d'une manière essentielle, à la condition d'être toujours convergentes pour toutes les valeurs réelles ou imaginaires de x, je me suis pro posé de déterminer a_m et b_m de telle manière que le quotient

$$\frac{\Sigma a_m e^{2m\frac{i\pi x}{}}}{\Sigma b_m e^{2m\frac{i\pi x}{a}}},$$

admette une autre période b. En posant $q = e^{i\pi\frac{b}{a}}$, cela conduit a l'égalité

$$\frac{\Sigma a_m e^{2m\frac{i\pi x}{a}}}{\Sigma b_m e^{2m\frac{i\pi x}{a}}} = \frac{\Sigma a_m q^{2m} e^{2m\frac{i\pi x}{a}}}{\Sigma b_m q^{2m} e^{2m\frac{i\pi x}{a}}}.$$

Chassant les dénominateurs, on trouve pour les coefficients d'une même exponentielle $e^{2\mu\frac{i\pi x}{a}}$, dans le premier membre et dans le second, respectivement les deux séries

$$\sum_{-\infty}^{+\infty}{}_m a_m b_{\mu-m} q^{2(\mu-m)} \quad \text{et} \quad \sum_{-\infty}^{+\infty}{}_m a_m b_{\mu-m} q^{2m}.$$

Or la manière la plus simple d'arriver à les rendre égales, consiste à les rendre identiques en sorte qu'un terme quelconque de l'une, tel que $a_m b_{\mu-m} q^{2(\mu-m)}$, ait son égal $a_n b_{\mu-n} q^{2n}$, dans l'autre.

Faisons donc, et cela pour toute valeur de l'entier μ,

$$a_m b_{\mu-m} q^{2(\mu-m)} = a_n b_{\mu-n} q^{2n},$$

et concevons que n soit exprimé en m de manière à produire le serie des nombres entiers, lorsque m prend lui-même toutes ses valeurs. On réalisera cette circonstance en prenant $n = m + k$, k étant un entier quelconque, l'équation précédente pourra alors s'écrire

$$\frac{a_m}{a_{m+k}} q^{-2(m+k)} = \frac{b_{\mu-(m+k)}}{b_{\mu-m}} q^{-2(\mu-m)},$$

et comme μ est quelconque, si l'on fait $\mu - m - k = m'$, m' sera un nombre entier variable entièrement indépendant de m, or il vient ainsi

$$\frac{a_m}{a_{m+k}} q^{-2(m+k)} = \frac{b_{m'}}{b_{m'+k}} q^{-2(m'+k)},$$

Sous cette forme, ou voit que chaque membre est une quantité constante ce qui conduit aux égalités

$$\frac{a_m}{a_{m+k}}\, q^{-2(m+k)} = \text{const.} \quad \frac{b_m}{b_{m+k}}\, q^{-2(m+k)} = \text{const.}$$

Donc a_m et b_m, dépendent de la même équation,

$$\frac{z_m}{z_{m+k}}\, q^{-2(m+k)} = \text{const.}$$

qu'on peut encore écrire sous la forme

$$z_m = z_{m+k}\, q^{2(m+k)+\alpha},$$

en changeant de constante, la solution générale est

$$z_m = \Pi(m)\, q^{-\frac{m^2}{k} - \alpha m}$$

la fonction $\Pi(m)$ étant assujétie à la condition suivante ;

$$\Pi(m+k) = \Pi(m).$$

Nous voici donc arrivés à cette forme analytique d' une fonction $\phi(x)$ aux périodes a et b savoir

$$\phi(x) = \frac{\Sigma\Pi(m)\, q^{-\frac{m^2}{k} - \alpha m}\, e^{2m\frac{i\pi x}{a}}}{\Sigma\Phi(m)\, q^{-\frac{m^2}{k} - \alpha m}\, e^{2m\frac{i\pi x}{a}}},$$

ou la condition de convergence prouve immédiatement, qu' en supposant $\frac{b}{a} = \omega + \tau\omega'$, le nombre entier k doit être du signe de ω' sa valeur absolue reste d'ailleurs arbitraire. En désignant par $\Theta(x)$ le numérateur ou le dénominateur, on trouvera en suite

$$\Theta(x+a) = \Theta(x), \quad \Theta(x+b) = \Theta(x)\, e^{\frac{ki\pi}{a}\left\{2x + (1-\alpha)b\right\}},$$

et de ces équations qui ne comportent d' arbitraire que la fonction périodique $\Pi(m)$, se déduisent toutes les propriétes caractéristiques des fonctions elliptiques, sans qu' il soit nécessaire pour cela d' établir préalablement, l' équation différentielle, tous les résultats déja connus, pouvant se démontrer indépendamment les uns des autres.

ON Γa, ESPECIALLY WHEN a IS NEGATIVE.

By FRANCIS W. NEWMAN.

THE importance of bringing the function Γa, or $\int_0^\infty e^{-x} x^{a-1} dx$, within the elementary departments of Analysis, is acknowledged. In so doing, the more difficult steps are found to be the establishment of the equation

$$F(a, b) \text{ or } \int_0^1 x^{a-1}(1-x)^{b-1}\, dx = \frac{\Gamma a \,.\, \Gamma b}{\Gamma(a+b)},$$

and the extension of Γa to the case of a negative. It is proposed here to indicate the order and mode of investigation, by which all may be effected, without involving any process at which the most cautious learner could justly hesitate.

We commence with the *Primary Properties of F*. We suppose a and b positive; then F is finite.

From the first principles of integration, it is obvious to prove

$$F(a, b) = F(b, a) \ldots\ldots\ldots\ldots\ldots (1),$$

$$F(a', b) < F(a, b), \text{ if } a' > a \ldots\ldots\ldots (2),$$

$$F(a, b) = \left(1 + \frac{b}{a}\right) F(a+1, b) \ldots\ldots\ldots (3).$$

So far, all reasoners, from Euler downward, have proceeded alike. Observe now, that if $b \div a$ is an evanescent fraction, we have from the last, that $F(a, b) = F(a+1, b)$,—or the limit of their ratio is 1, when $b \div a$ perpetually lessens. But (for conciseness merely) we may here dispense with the latter mode of expression.

It follows, that if r is any finite integer, and $(b \div a)$ evanescent, we have $F(a, b) = F(a+1, b) = \ldots\ldots = F(a+r, b)$.

Also, if δ is a proper fraction, we see by (2) that $F(a+r+\delta, b)$ is between $F(a+r, b)$ and $F(a+r+1, b)$. Put $r+\delta$ = any finite number α, not necessarily integer, and we find

$$F(a, b) = F(a+\alpha, b), \quad \text{if } \frac{b}{a} = 0 \text{ and } \alpha \text{ finite} \ldots\ldots (4).$$

We proceed to develop $F(a, b)$ in a series of factors.

Repeatedly applying equation (3), we get

$$F(a, b) = \frac{a+b}{a} \,.\, \frac{1+a+b}{1+a} \ldots\ldots \frac{n-1+a+b}{n-1+a} \,.\, F(n+a, b).$$

Write 1 for a, $n-1$ for n, in the last; then

$$F(1, b) = \frac{1+b}{1} \,.\, \frac{2+b}{2} \ldots\ldots \frac{n-1+b}{n-1}\, F(n, b).$$

Divide the former by the latter, observing that $F(1, b) = b^{-1}$;

$$\therefore F(a, b) = \frac{a+b}{ab} \cdot \frac{1 \,.\, 1+a+b}{1+a \,.\, 1+b} \cdot \frac{2 \,.\, 2+a+b}{2+a \,.\, 2+b} \text{. to } n \text{ factors}$$

$$\times \frac{F(n+a, b)}{F(n, b)}.$$

Now in equation (4) write n for a, and a for a; make $n = \infty$, and we find $F(n, b) = F(n + a, b)$, provided that a is finite: hence

$$F(a, b) = \frac{a+b}{ab} \cdot \frac{1+(a+b)}{1+a \,.\, 1+b} \cdot \frac{1+\frac{1}{2}(a+b)}{1+\frac{1}{2}a \,.\, 1+\frac{1}{2}b} \cdot \frac{1+\frac{1}{3}(a+b)}{1+\frac{1}{3}a \,.\, 1+\frac{1}{3}b}$$

$$\ldots\ldots \text{\&c.} \ldots\ldots (5).$$

Examine this result, and it appears that if we venture to write

$$\psi(a) \text{ for } a(1+a)(1+\tfrac{1}{2}a)(1+\tfrac{1}{3}a) \text{ \&c.} \ldots\ldots$$

we shall have $F(a, b) = \psi(a+b) \div (\psi a, \psi b)$; which resolves F, a function of two variables, into ψ, a function of one. But a little inspection shews ψ to be infinite; since

$$\log \psi a = \log a + \log(1+a) + \log(1+\tfrac{1}{2}a) + \log(1+\tfrac{1}{3}a) + \text{\&c.}$$

$$= \log a + (1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \text{\&c.})\, a - \tfrac{1}{2} S_2 a^2 + \tfrac{1}{3} S_3 a^3 - \text{\&c.} \ldots\ldots$$

by developing, when a is < 1. Now as the coefficient of a is infinite, so is ψ. Nevertheless, that suggests another assumption, free from this inconvenience. Suppose

$$\psi a = a \,.\, \frac{1+a}{e^a} \cdot \frac{1+\frac{1}{2}a}{e^{\frac{1}{2}a}} \cdot \frac{1+\frac{1}{3}a}{e^{\frac{1}{3}a}} \text{ . \&c. \&c.} \ldots\ldots (6),$$

$$\therefore \log \psi a = \log a - \tfrac{1}{2} S_2 a^2 + \tfrac{1}{3} S_3 a^3 - \tfrac{1}{4} S_4 a^4 + \text{\&c.} \ldots\ldots$$

which is finite. And since $e^{r+s} = e^r \,.\, e^s$, we have quite as much as before

$$F(a, b) = \frac{\psi(a+b)}{\psi a \,.\, \psi b} \ldots\ldots\ldots (7).$$

Change a into $(a + n)$ in (7);

$$\therefore F(a+n, b) = \psi(a+n+b) \div \{\psi(a+n) \,.\, \psi b\},$$

whence we deduce $\dfrac{F(a, b)}{F(a+n, b)} = \dfrac{\psi(a+b) \,.\, \psi(a+n)}{\psi a \,.\, \psi(a+n+b)}$.

And since the right-hand member is not affected by exchanging n with b, Euler's well-known result follows:

$$\frac{F(a, b)}{F(a+n, b)} = \frac{F(a, n)}{F(a+b, n)}.$$

whence by equation (1) we have

$$F(a, b) = \frac{F(n, a) \cdot F(n + a, b)}{F(n, a + b)}.$$

Make n infinite in comparison with a and b, and observe that by (4) we then find $F(n + a, b) = F(n, b)$: consequently we get

$$F(a, b) = \frac{F(n, a) \cdot F(n, b)}{F(n, a + b)}, \text{ when } n = \infty \dots . (8).$$

This equation has a close analogy with (7), and leads us to examine the nature of the function $F(n, b)$ when $n = \infty$.

Put $x = n^{-1} y$; therefore

$$F(n + 1, a) = \int_0^1 (1 - x)^n x^{a-1} dx = \int_0^n (1 - n^{-1} y)^n n^{-a} y^{a-1} dy$$
$$= n^{-a} \cdot \int_0^\infty e^{-y} y^{a-1} dy.$$

Denoting therefore the last integral by Γa, we obtain from equation (8) by mere substitution,

$$F(a, b) = \frac{\Gamma a \cdot \Gamma b}{\Gamma (a + b)} \dots\dots\dots\dots\dots (9).$$

Comparing (7) and (9), it occurs to inquire, what is the relation between ψ and Γ. The one *might* be the reciprocal of the other, as far as these two equations are concerned. Put then $\Gamma a = \chi a \cdot (\psi a)^{-1}$, where χ is an unknown function. Substitute this value for Γ in (9) and simplify the result by (7). There remains $\chi (a + b) = \chi a \cdot \chi b$; which can be solved either by differentiating or by still more elementary methods, and gives $\chi a = e^{2\gamma a}$, where γ is an unknown constant.

Hence $$\Gamma a = \frac{e^{-\gamma a}}{a} \cdot \frac{e^a}{1 + a} \cdot \frac{e^{\frac{1}{2}a}}{1 + \frac{1}{2} a} \cdot \frac{e^{\frac{1}{3}a}}{1 + \frac{1}{3} a} \cdot \&c. . (10),$$

and when a^2 is < 1,

$$\log \Gamma a = - \log a - \gamma a + \{a - \log (1 + a)\}$$
$$+ \{\tfrac{1}{2} a - \log (1 + \tfrac{1}{2} a)\} + \&c. \dots$$
$$= - \log a - \gamma a + \tfrac{1}{2} S_2 a^2 - \tfrac{1}{3} S_3 a^3 + \tfrac{1}{4} S_4 a^4 - \&c. \dots (11).$$

Proceeding to the integral $\int_0^\infty e^{-x} x^{a-1} dx$, it is obvious from the first principles of integration that

$$\Gamma (a + 1) = a \Gamma a;\ \Gamma 1 = 1;\ \Gamma n = 1 . 2 . 3 . . (n - 1) \dots (12).$$

so that we readily find in (11) that

$$\left.\begin{aligned}\gamma &= (1 - \log 2) + (\tfrac{1}{2} - \log \tfrac{3}{2}) + (\tfrac{1}{3} - \log \tfrac{4}{3}) + \&c.\\ &= \tfrac{1}{2} S_2 - \tfrac{1}{3} S_3 + \tfrac{1}{4} S_4 - \tfrac{1}{5} S_5 + \&c.\end{aligned}\right\}\ldots\ldots(13).$$

Indeed from (11) it is easy to find $(1 - \gamma)$ by still more rapid convergence. But having sufficiently determined γ, let us take up the consideration of Γ from a new point of view. Namely, let us assume the equation (10) as the *definition** of Γ; then we know this to be identical with $\Gamma a = \int_0^\infty e^{-x} x^{a-1}\, dx$, as long as a is positive: so that all inferences drawn from the latter still hold good, under that limitation. Nevertheless, when a is negative, it is not the less true that $\Gamma(a + 1) = a\,\Gamma a$; which may be thus shewn.

Since $$\Gamma a = \frac{e^{-\gamma a}}{a} \cdot \frac{e^a}{1 + a} \cdot \frac{e^{\frac{1}{2}a}}{1 + \frac{1}{2}a} \cdot \frac{e^{\frac{1}{3}a}}{1 + \frac{1}{3}a} \cdot \&c. \ldots\ldots$$

change a into $-a$, and multiply by the result

$$\Gamma a \,.\, \Gamma(-a) = -a^{-2}\left\{(1 - a^2)(1 - \tfrac{1}{4}a^2)(1 - \tfrac{1}{9}a^2)\ldots\right\}^{-1}$$

$$= \frac{-\pi a^{-1}}{\sin . \pi a}.$$

Suppose a positive;

therefore $$\Gamma(-a) = \frac{-\pi}{\sin \pi a \,.\, \Gamma(1 + a)} \ldots\ldots (14).$$

Write $(a - 1)$ for a; then

$$\Gamma(1 - a) = \frac{\pi}{\sin \pi a \,.\, \Gamma a} = -a \,.\, \Gamma(-a):$$

and the last is an extension of $\Gamma(1 + a) = a\,\Gamma a$ to the case of a negative.

Then $\Gamma(1 + a)\,\Gamma(-a) = -\pi \div \sin \pi a$, for *all* values of a;

or $$\Gamma(1 - a) \,.\, \Gamma a = \pi \div \sin \pi a \ldots\ldots\ldots\ldots (15).$$

Thus the Complementary Equation is demonstrated for all values of a: and all other properties of Γ readily flow out of (11) (12) and (15).

* Then even if a is imaginary, it can be strictly proved that $\Gamma(1 + a) = a\,\Gamma a$, by taking, first, n factors of the series, and afterwards making $n = \infty$: also that $\Gamma(1 + a) \,.\, \Gamma(1 - a) = \pi a \div \sin \pi a$.

TO DEVELOP $(\cos x)^a$ IN A SERIES OF COSINES FOR ALL VALUES OF a.

By Francis W. Newman.

Let $X = \log(2\cos x) = \cos 2x - \frac{1}{2}\cos 4x + \frac{1}{3}\cos 6x - \&c..$ when x is between $\pm\frac{1}{2}\pi$; therefore

$$(2\cos x)^a = 1 + \frac{a}{1}X + \frac{a^2}{1.2}X^2 + \frac{a^3}{1.2.3}X^3 + \&c...$$

Now X^n contains nothing but terms made up of factors $(\cos 2mx)^r$, where m and r are positive integers; hence X^n is expressible as $a' + \Sigma b\cos 2nx$. Consequently, while x is between $\pm\frac{1}{2}\pi$, we may assume

$$(\cos x)^a = A + A_1\cos 2x + A_2\cos 4x + A_3\cos 6x + \&c.\dots.(1),$$

where $A, A_1, A_2 \dots$ are unknown functions of a.

Put $y = (\cos x)^a$, therefore $ay\sin x + \dfrac{dy}{dx}\cos x = 0$. By substituting as usual for y and its differential coefficient in terms of x, and reducing to linear sines, we find that the differential equation will be satisfied, if

$$2aA = (a+2)A_1;\quad (a-2)A_1 = (a+4)A_2;$$
$$(a-4)A_2 = (a+6)A_3;\ \&c.\dots\dots.(2),$$

so that $A_1, A_2, A_3 \dots$ are known multiples of A; and we get

$$(\cos x)^a = A.\left\{1 + \frac{2a\cos 2x}{a+2} + \frac{2a.a-2.\cos 4x}{a+2.a+4} + \frac{2a.a-2.a-4.\cos 6x}{a+2.a+4.a+6} + \&c.\dots\right\}\dots.(3).$$

To find A, multiply the series by dx, and integrate from $x = 0$ to $x = \frac{1}{2}\pi$, observing that

$$\int_0^{\frac{1}{2}\pi}\cos 2nx\,dx = 0,\quad \text{and}\quad \int_0^{\frac{1}{2}\pi}(\cos x)^a\,dx$$

is found by making $\cos x = \sqrt{z}$, to be

$$= \tfrac{1}{2}F\left\{\tfrac{1}{2}(a+1), \tfrac{1}{2}\right\} = \frac{\Gamma.\frac{1}{2}(a+1).\Gamma\frac{1}{2}}{2.\Gamma.\frac{1}{2}(a+2)};\ \text{also}\ \Gamma\tfrac{1}{2} = \sqrt{\pi}.$$

If then we can assume the entire series, after the first term, to vanish, when every term vanishes, we shall get

$$\frac{\sqrt{\pi}.\Gamma.\frac{1}{2}(a+1)}{2.\Gamma.\frac{1}{2}(a+2)} = A.\tfrac{1}{2}\pi;\ \text{or}\ A = \frac{\Gamma.\frac{1}{2}(a+1)}{\sqrt{\pi}\Gamma.\frac{1}{2}(a+2)}\dots.(4).$$

To test the assumption, observe that

$$(2n + a + 2)\, A_{n+1} = -\,(2n - a)\, A_n\,;$$

so that for very large values of n, A_{n+1} and A_n are of opposite sign; and tend towards equality, whether a be positive or negative. Hence the latter part of the series tends to coincidence with

$$y = A_n\,(\cos 2nx - \cos (2n + 2)\, x + \cos (2n + 4)\, x - \&\text{c.}\ldots).$$

Now

$$\int_0 y\,dx = \tfrac{1}{2} A_n \left(\frac{\sin 2nx}{n} - \frac{\sin (2n+2)x}{n+1} + \frac{\sin (2n+4)x}{n+2} - \&\text{c.} \right)$$

The series here is known to converge, and vanishes when $x = \frac{1}{2}\pi$: and by taking n a large finite number, we may make our series agree as nearly as we please with this; and unless A_n increases so rapidly with n as to give a finite value to the product, the problem is solved. Now when a is positive, the coefficients A, A_1, A_2, A_3. . . . diminish; so that A_n is finite or zero. If however a be negative, we must inquire farther.

$$\text{Now } A_n = (-1)^{n-1}.\, 2A\,.\, \frac{a}{2+a} \cdot \frac{2-a}{4+a} \cdot \frac{4-a}{6+a} \cdots\cdots \frac{2n-2-a}{2n+a}$$

$$= (-1)^{n-1}.\, \frac{2Aa}{2n+a} \cdot \frac{2-a}{2+a} \cdot \frac{4-a}{4+a} \cdots\cdots \frac{2n-2-a}{2n-2+a}.$$

This may be computed from the properties of Γ, when n is very large. For we have, as approximate equations (*true* only when $n = \infty$),

$$\Gamma a = \frac{e^{-\gamma a}}{a} \cdot \frac{e^{a}}{1+a} \cdot \frac{e^{\frac{1}{2}a}}{1+\frac{1}{2}a} \cdots\cdots \frac{e^{n^{-1}a}}{1+n^{-1}a}:$$

$$\frac{\sin \pi a}{\pi a} = (1 - a^2)\left(1 - \frac{a^2}{4}\right) \cdots\cdots \left(1 - \frac{a^2}{n^2}\right):$$

Multiply the latter by the square of the former;

$$(\Gamma a)^2.\, \frac{\sin \pi a}{\pi a} = \frac{e^{-2\gamma a}}{a^2}\, e^{2a(1+2^{-1}+\ldots+n^{-1})}.\, \frac{1-a}{1+a} \cdot \frac{2-a}{2+a} \cdots\cdots \frac{n-a}{n+a}.$$

When n is very large $(1 + 2^{-1} + 3^{-1} + \ldots + n^{-1})$ nearly $= \gamma + \log n$, so that if we write $\frac{1}{2}a$ for a, we get more simply

$$(\Gamma \tfrac{1}{2}a)^2\, \frac{\sin (\frac{1}{2}\pi a)}{\frac{1}{2}\pi a} = \frac{4n^a}{a^2} \cdot \frac{2-a}{2+a} \cdot \frac{4-a}{4+a} \cdots\cdots \frac{2n-a}{2n+a};$$

$$\text{whence} \quad A_n = (-1)^{n-1}.\, \frac{Aa^2\pi^{-1}n^{-a}}{2n-a}\,.\,(\Gamma \tfrac{1}{2}a)^2 \sin \tfrac{1}{2}\pi a,$$

which is infinite when a is negative, and $n = \infty$.

Nevertheless let $(a+1)$ be positive, and $= b$, which is then between 0 and 1; therefore

$$A_n = (-1)^n\, 2A \,.\, \frac{1-b}{1+b} \,.\, \frac{3-b}{3+b} \,.\, \frac{5-b}{5+b} \,\ldots\, \frac{2n-1-b}{2n-1+b},$$

from which we see that $A, A_1, A_2 \ldots A_n$ is still a decreasing series; consequently we can trust our conclusion, provided that a is either positive, or, if negative, less than 1.

Thus, provided that $(1+a)$ is positive, and x is between $\pm \frac{1}{2}\pi$, we have

$$(\cos x)^a = \frac{1}{\sqrt{\pi}} \,.\, \frac{\Gamma . \frac{1}{2}(a+1)}{\Gamma . \frac{1}{2}(a+2)} \,.\, \left\{ 1 + \frac{2a \cos 2x}{2+a} - \frac{2a \,.\, 2-a \,.\, \cos 4x}{2+a \,.\, 4+a} \right.$$

$$\left. + \frac{2a \,.\, 2-a \,.\, 4-a \,.\, \cos 6x}{2+a \,.\, 4+a \,.\, 6+a} - \&c. \right\}.$$

which includes the well-known development of $(\cos x)^{2n}$; the series terminating when $a = 2n$.

When $a = 2n-1$, $\Gamma . \frac{1}{2}(a+1)$ and $\Gamma . \frac{1}{2}(a+2)$ may be simplified, and we get

$$A = \frac{2}{\pi} \,.\, \frac{2 \,.\, 4 \ldots . 2n-2}{3 \,.\, 5 \ldots . 2n-1}.$$

In particular, if $a = 1$, $n = 1$,

$$\tfrac{1}{4}\pi \cos x = \tfrac{1}{2} + \frac{\cos 2x}{1.3} - \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} - \&c. \ldots .$$

When a reaches the limit -1, A becomes infinite, and we have

$$(\cos x)^{-1} = A\,(1 - 2\cos 2x + 2\cos 4x - 2\cos 6x + \&c. \ldots .),$$

where the series is indeterminate. We may perhaps conclude that the development in this form is impossible, when $(-a)$ exceeds 1.

ON THE DETERMINATION OF THE MODULUS OF ELASTICITY OF A ROD OF ANY MATERIAL, BY MEANS OF ITS MUSICAL NOTE.

By Andrew Bell.

It is proposed in this paper to determine the modulus of elasticity of any material, by means of the musical note obtained from a rod of the material. The modulus being

determined, it will of course thence be possible to ascertain the weight a column of the material can support before beginning to bend, and other elements dependent on the modulus.

For short rods, suppose a foot long, the longitudinal note is generally very acute; and as the ear cannot with great precision distinguish between notes near either extremity of the scale, and the transversal notes are much graver, the fundamental note, that is, the gravest note when the rod is fixed at one end and vibrates transversely without a node, could be observed and from it the gravest longitudinal note can be deduced by means of the relation between them expressed by the equation*

$$n = 7.12164 \frac{h}{l} n_1,$$

in which n and n_1 are these notes respectively, l the length, and h a quantity dependent on the normal section and which is generally much less than l, and n therefore much less than n_1.

The pitch of a note not very near either end of the scale can be determined to within a musical comma, and its value therefore is known to one 80th part, from which, as will be seen afterwards, the modulus can be found to within one 40th part; a limit of error at least five times less than that of the ordinary method of observing the modulus, even when the spars experimented on are similar and equal contiguous portions of the same beam, as may be seen by inspecting the tables on this subject (see Barlow's *Essay on the Strength of Materials*); if it be understood that such spars have the same or very nearly the same modulus, which is exceedingly probable.

In the equation of longitudinal vibration

$$\frac{d^2u}{dt^2} = a^2 \frac{d^2u}{dx^2};$$

the constant a is known to express the velocity of propagation of a wave; and on it therefore the pitch of the note will entirely depend for rods of the same length.

The longitudinal notes of a rod and a column of air of equal length will necessarily be proportional to the velocities of propagation of waves through them, and as the velocity through air is well known both from theory

* See Poisson, *Mécanique* II, 391.

and experiment; this relation therefore will determine the velocity through the material when the notes are observed.

The theoretical expression for the constant a is found to involve the modulus of elasticity*; that is, the length of a prism of the material under consideration, whose weight would be capable of distending another prism of it of the same cross section, to double its length, supposing it capable of this amount of uniform distention; or, in other words, such a length that the weight of its 1000th part would distend the prism one 1000th part of its length: or, in reference to a fluid, the modulus is such a height of it, considered homogeneous, that one 1000th part of it would compress any portion of it a 1000th part considering the compression uniform; or, in the case of elastic fluids the modulus is that height of the fluid considered homogeneous that produces the pressure to which the fluid is subjected. In the case therefore of atmospheric air, the modulus is just the height of what is termed the equivalent homogeneous atmosphere; and the velocity of propagation in air, were there no disengagement of free caloric by the undulatory condensations, would just be, as is well known, that due to half the height of its modulus.

The same relation as that last stated, subsists between this velocity and the modulus in the case of *solid* materials. For (Poisson, *Mécanique* II, 319)

$$a^2 = \frac{glq}{p},$$

where l and p are the length and weight of the rod, and q is the weight of the modulus, as appears from the relation

$$q = \frac{\Delta}{\delta},$$

in which Δ is the tension producing a fractional dilatation

* This is the original definition of the modulus of elasticity. The E now adopted in some engineering researches is the *weight in pounds* of this modulus, when the cross section is a square inch, and might more conveniently be denoted by W. The E used in some other engineering works, as in Barlow's "*Strength and Stress of Timber*," is just four times the former E. The former E or W is the weight that would distend a prismatic bar, having a transverse section of one square inch, to double its length, and is therefore related to M by the equation

$$M = 2304\,\frac{E}{G} = 2304\,\frac{W}{G},$$

G being the specific gravity of the material, that of water being 1000.

denoted by δ or an absolute distention expressed by δl; and as

$$\frac{q}{l} = \frac{\Delta}{\delta l},$$

therefore l is the quantity of distension for the tension q; which is therefore the weight of the modulus. Consequently if M be the modulus,

$$\frac{p}{q} = \frac{l}{M} \text{ and } M = \frac{lq}{p};$$

hence also $$a^2 = gM,$$

and a therefore is the velocity due to a vertical descent equal to $\frac{1}{2}M$.

A formula stating a relation between the note, the weight of the modulus, and the length and weight of the rod, (Poisson, *Mécanique* II, 321) might be adopted for determining the modulus; but it will be better to deduce a simpler and more interesting and more general relation, entirely independent of the elements of the rod and involving only the ratio of the velocities of propagation in the atmosphere and in the material and the modulus of both; or, in other words, to derive a formula involving only what may be termed statical and dynamical constants of the atmosphere and the material; and this formula will be more especially convenient as a very skilful experimentalist in this subject, Chladni, has given the results of his experiments on longitudinal vibrations in terms of that ratio. The proposed formula may be deduced thus: the corresponding notes, as the gravest for instance, produced by a cylinder of air in a tube and by the longitudinal vibrations of a prism of any material of the same length, the former being open, and the latter either free or fixed, at both ends; are directly as the velocities of propagation, which again are as the square roots of the moduli; these velocities therefore being known and also the modulus of the atmosphere, that of the given material can then be found.

Let a', M' denote this velocity and the modulus for the atmosphere, and a, M the same elements for the given material; then, since

$$a'^2 = gM' \text{ and } a^2 = gM,$$

therefore $$M = \left(\frac{a}{a'}\right)^2 M' = r^2 M' \text{ suppose.}$$

The results of Chladni's experiments (Biot, *Traité de Physique* II, 76) give the ratios of the notes for columns

of air and rods of the same length for a variety of materials, glass, wood, and metal, and these are the same as the ratios of a and a'.

The velocity a' is however not that due to $\frac{1}{2}M'$, for, as is well known, the momentary disengagement of caloric by the undulatory condensations, notwithstanding the alternating rarefactions, increases the latter velocity one-fifth. Assuming the temperature of the air at 56° and its pressure 30 in., the value of a' is 1120 and of M' very nearly 27500 ft.; hence,

$$4M = 110000r^2.$$

The undulation in passing along the prism will undoubtedly cause an alternate lateral distention and contraction of the prismatic element, which will cause the alternate increase and diminution of density to be comparatively less than in a confined or a free mass of air; still the alternate compression and dilatation will, as in the case of elastic fluids, certainly cause some disengagement of free caloric both in organic and inorganic materials, and thus the velocity of propagation of a disturbance will be in some measure accelerated.*

Since the limit of error in estimating the musical note in the same rod is about one 80th of an interval, the limit of error produced on a with the same rod, is $\frac{1}{80}a$, and on a^2 it is

$$= a^2(1 + \tfrac{1}{80})^2 - a^2 = \tfrac{1}{40}a^2 \text{ nearly};$$

hence the limit of error on M by experimenting with a single rod is only $\frac{1}{40}M$.

As a practical illustration of the formula, some of Chladni's results may be taken. Thus,

for oak $r = 10\frac{2}{3}$ therefore $M = 110000\left(\frac{r}{2}\right)^2 = 110000\,(5\frac{1}{3})^2 = 3130000$;

for elm $r = 14\frac{2}{5}$, hence, $M = 5702400$.

6, *Nelson Street, Edinburgh, Oct.* 12, 1847.

[* Our ignorance of the amount of this effect, and our consequent inability to make the necessary correction for it, are such that the practical application suggested in this paper, cannot, in the present state of science, be considered as likely to lead to very accurate results.]

ON SYMBOLICAL GEOMETRY.

By Sir William Rowan Hamilton.

[*Continued from* p. 209, Vol. II.]

Condition of Concircularity, resumed. New Equation of a Cyclic Cone.

25. The equation (150) of *homocyclicism*, or of *concircularity*, which was assigned in the 20th article, and which expresses the condition requisite in order that four straight lines in space, a, b, c, d, diverging from one common point O, as from an origin, may terminate in four other points A, B, C, D, which shall all be contained on the circumference of one common circle, may also, by (149), be put under the form

$$\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} = x\frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}} \ldots\ldots\ldots\ldots (173),$$

where x is a scalar coefficient. It gives therefore the two following separate equations, one between scalars, and the other between vectors:

$$\mathrm{S}\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} = x\mathrm{S}\frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}};\quad \mathrm{V}\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} = x\mathrm{V}\frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}} \ldots (174);$$

of which the latter is only another way of writing the equation (151). If then we agree to use, for conciseness, a new characteristic of operation, $\frac{\mathrm{V}}{\mathrm{S}}$, of which the effect on any geometrical fraction, to the symbol of which it is prefixed, shall be defined by the formula

$$\frac{\mathrm{V}}{\mathrm{S}}\cdot\frac{\mathrm{b}}{\mathrm{a}} = \mathrm{V}\frac{\mathrm{b}}{\mathrm{a}} \div \mathrm{S}\frac{\mathrm{b}}{\mathrm{a}} \ldots\ldots\ldots\ldots (175);$$

so that this new characteristic $\frac{\mathrm{V}}{\mathrm{S}}$, which (it must be observed) *is not a distributive symbol*, is to be considered as directing to *divide the vector by the scalar part* of the geometrical fraction on which it operates; we shall then have, as a consequence of (173), this other form of the *equation of concircularity:*

$$\frac{\mathrm{V}}{\mathrm{S}}\cdot\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} = \frac{\mathrm{V}}{\mathrm{S}}\cdot\frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}} \ldots\ldots\ldots (176).$$

Conversely we can return from this latter form (176) to the equation (173); for if we observe that, in the present system of symbolical geometry, *every geometrical fraction is equal to*

the sum of its own scalar and vector parts, so that we may write generally (see article 7),

$$\mathrm{S}\,\frac{\mathrm{b}}{\mathrm{a}} + \mathrm{V}\,\frac{\mathrm{b}}{\mathrm{a}} = \mathrm{V}\,\frac{\mathrm{b}}{\mathrm{a}} + \mathrm{S}\,\frac{\mathrm{b}}{\mathrm{a}} = \frac{\mathrm{b}}{\mathrm{a}} \ldots\ldots\ldots (177),$$

or more concisely, $\mathrm{S} + \mathrm{V} = \mathrm{V} + \mathrm{S} = 1 \ldots\ldots\ldots (178)$;

and, if we add the identity,

$$\mathrm{S}\,\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} \div \mathrm{S}\,\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} = \mathrm{S}\,\frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}} \div \mathrm{S}\,\frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}} \ldots (179),$$

of which each member is equal to unity, to the equation (176), attending to the definition (175) of the new characteristic lately introduced, we are conducted to this other formula,

$$\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} \div \mathrm{S}\,\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} = \frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}} \div \mathrm{S}\,\frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}} \ldots (180);$$

which allows us to write also

$$\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} \div \frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}} = \mathrm{S}\,\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} \div \mathrm{S}\,\frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}} \ldots (181),$$

where the second member, being the quotient of two scalars, is itself another scalar, which may be denoted by x; and thus the equation (173) may be obtained anew, as a consequence of the equation (176). We may therefore also deduce from the last-mentioned equation the following form,

$$\frac{\mathrm{c}-\mathrm{d}}{\mathrm{d}-\mathrm{a}} = x\,\frac{\mathrm{c}-\mathrm{b}}{\mathrm{b}-\mathrm{a}} \ldots\ldots\ldots (182);$$

and thence also, by a new elimination of the scalar coefficient x, performed in the same manner as before, may derive this other form,

$$\frac{\mathrm{V}}{\mathrm{S}}\cdot\frac{\mathrm{c}-\mathrm{d}}{\mathrm{d}-\mathrm{a}} = \frac{\mathrm{V}}{\mathrm{S}}\cdot\frac{\mathrm{c}-\mathrm{b}}{\mathrm{b}-\mathrm{a}} \ldots\ldots\ldots (183).$$

Indeed, the geometrical signification of the condition (176) shews easily that we may in any manner transpose, in that condition, the symbols a, b, c, d; since if, *before* such a transposition, those symbols denoted four diverging straight lines (not generally in one common plane), which terminate on the circumference of one common circle, then *after* this transposition they must still denote four such diverging lines. We may therefore interchange the symbols a and c, in the condition (176), which will thus become

$$\frac{\mathrm{V}}{\mathrm{S}}\cdot\frac{\mathrm{b}-\mathrm{c}}{\mathrm{a}-\mathrm{b}} = \frac{\mathrm{V}}{\mathrm{S}}\cdot\frac{\mathrm{d}-\mathrm{c}}{\mathrm{a}-\mathrm{d}} \ldots\ldots\ldots (184);$$

but also, as in ordinary algebra, we have here,

$$\frac{\mathrm{b}-\mathrm{c}}{\mathrm{a}-\mathrm{b}}=\frac{\mathrm{c}-\mathrm{b}}{\mathrm{b}-\mathrm{a}}; \quad \frac{\mathrm{d}-\mathrm{c}}{\mathrm{a}-\mathrm{d}}=\frac{\mathrm{c}-\mathrm{d}}{\mathrm{d}-\mathrm{a}} \ldots\ldots (185);$$

the equation (183) might therefore have been in this other way deduced from the equation (176), as another form of the same condition of concircularity: and it is obvious that several other forms of the same condition may be obtained in a similar way.

26. From the fundamental importance of the *circle* in geometry, it is easy to foresee that these various forms of the condition of concircularity must admit of a great number of geometrical applications, besides those which have already been given in some of the preceding articles of this essay on Symbolical Geometry. For example, we may derive in a new way a solution of the problem proposed at the beginning of the 20th article, by conceiving that the symbols a, b, c denote three given sides of a *cyclic cone,* extending from the vertex to some given plane which is parallel to that one of the two *cyclic planes* which in the problem is supposed to be given; for then the equation (183) may be employed to express that the variable line d is a fourth side of the same cyclic cone, drawn from the same vertex as an origin, and bounded by the same given plane, or terminating on the same circumference, or circular base of the cone, as the three given sides, a, b, c. Or we may change the symbol d to another symbol of the form xx, and may conceive that x denotes a variable side of the cone, still drawn as before from the vertex, but not now necessarily terminating on any one fixed plane, nor otherwise restricted as to its length; while x shall denote a scalar coefficient, or multiplier, so varying with the side or line x as to render the product-line xx a side of which the extremity is (like that of d) concircular with the given extremities of a, b, c; and we may express these conceptions and conditions by writing as the equation of the cone the following:

$$\frac{\mathrm{V}}{\mathrm{S}} \cdot \frac{\mathrm{c}-x\mathrm{x}}{x\mathrm{x}-\mathrm{a}}=\beta \ldots\ldots\ldots\ldots (186);$$

where β is a given geometrical fraction of the *vector* class, namely, that vector which is determined by the equation

$$\frac{\mathrm{V}}{\mathrm{S}} \cdot \frac{\mathrm{c}-\mathrm{b}}{\mathrm{b}-\mathrm{a}}=\beta \ldots\ldots\ldots\ldots (187).$$

The *index* $I\beta$ of this vector β is such that

$$I\beta \parallel I \frac{c - b}{b - a} \ldots\ldots\ldots\ldots (188);$$

it is therefore (by the principles of articles 7 and 10) a line perpendicular to each of the two lines represented by the two symbolical differences $c - b$, $b - a$, and therefore also perpendicular to the line denoted by their symbolic sum, $c - a$; so that we may establish the three formulæ,

$$I\beta \perp c - b; \quad I\beta \perp b - a; \quad I\beta \perp c - a \ldots (189),$$

and may say that $I\beta$ is a line *perpendicular to the plane* in which the three lines a, b, c all terminate. This constant index $I\beta$, connected with the equation (186) of the cyclic cone just now determined, as being the *index of the constant vector fraction* β, to which the first member of that equation is equal, is therefore perpendicular also to the given cyclic plane of the same cone, and may be regarded as a symbol for one of the two *cyclic normals* of that conical locus of the variable line x lately considered. In the particular case when the three given lines a, b, c are all *equally long*, so that the cyclic cone (186) becomes a *cone of revolution*, then the index $I\beta$, which had been generally a symbol for a cyclic normal, becomes a symbol for the *axis of revolution* of the cone. Other forms of equations of such cyclic and other cones will offer themselves when the principles of the present system of symbolical geometry shall have been more completely unfolded; but the forms just given will be found to be sufficient, when combined with some of the equations assigned in previous articles, to conduct to the solution of some interesting geometrical problems: to which class it will perhaps be permitted to refer the general determination of the *curvature of a spherical conic*, or the construction of the cone of revolution which *osculates* along a given side to a given cyclic cone.

Curvature of a Spherical Conic, or of a Cyclic Cone.

27. To treat this problem by a method which shall harmonise with the investigations of recent articles of this paper, let the symbols a', c', d'', be employed with the same significations as in article 24, so as to denote three equally long straight lines, of which a' is a trace of one cyclic plane on the other, while c' and d'' are the traces of a tangent plane on those two cyclic planes; and let c (still bisecting the angle between c' and d'') be still the equally long side of contact

of that tangent plane with the given cyclic cone. We shall then have, by (156), the symbolic analogy,

$$d'' : c :: c : c' \ldots\ldots\ldots\ldots\ldots (190),$$

which, on account of the supposed equality of the lengths of the lines c, c′, d″, gives also the two following formulæ, of parallelism and perpendicularity,

$$d'' + c' \parallel c; \quad d'' - c' \perp c \ldots\ldots\ldots (191);$$

of which indeed the former has been given already, as the first of the two formulæ (160). Conceive next that through the side of contact c we draw two secant planes, cutting the same sheet of the cone again in two known sides, c_1, c_2, and having for their known traces on the first cyclic plane (which contains the trace c′ of the tangent plane) the lines c_1', c_2', but for their traces on the second cyclic plane (or on that which contains d″) the lines d_1'', d_2''; these lines, $cc_1c_2c_1'c_2'd_1''d_2''$, being supposed to be all equally long. We may then write (in virtue of what has been shewn in former articles) at once the two new symbolic *analogies*,

$$d_1'' : c_1 :: c : c_1'; \quad d_2'' : c_2 :: c : c_2' \ldots (192);$$

the two new *parallelisms*,

$$d_1'' + c_1' \parallel c_1 + c; \quad d_2'' + c_2' \parallel c_2 + c \ldots (193);$$

and the two new *perpendicularities*,

$$d_1'' - c_1' \perp c_1 + c; \quad d_2'' - c_2' \perp c_2 + c \ldots (194):$$

we shall have also these two other formulæ of parallelism,

$$d_1'' - c_1' \parallel c_1 - c; \quad d_2'' - c_2' \parallel c_2 - c \ldots (195).$$

Now if we conceive a cone of revolution to contain upon one sheet the three equally long lines c, c_1, c_2, which are also (by the construction) three sides of one sheet of the given cyclic cone, we may (by the last article) represent a line in the direction of the *axis* of this cone of revolution by the symbol,

$$\mathrm{I}\,\frac{c_2 - c}{c - c_1} \ldots\ldots\ldots\ldots\ldots (196);$$

or by this other symbol, which denotes indeed a line having an opposite direction, but still one contained upon the indefinite axis of the same cone of revolution, if drawn from a point on that axis,

$$\mathrm{I}\,\frac{c_2 - c}{c_1 - c} \ldots\ldots\ldots\ldots\ldots (197).$$

On account of the parallelisms (195) we may substitute for the last symbol (197) this other of the same kind,

$$\mathrm{I}\,\frac{d_2'' - c_2'}{d_1'' - c_1'} \ldots\ldots\ldots\ldots\ldots \quad (198);$$

which expression, when we add to it another, which is a symbol of a null line (because in general the index of a scalar vanishes), namely the following,

$$\mathrm{I}\,\frac{c_1' - d_1''}{d_1'' - c_1'} = 0 \ldots\ldots\ldots\ldots (199),$$

takes easily this other form,

$$\mathrm{I}\,\frac{d_2'' - c_2'}{d_1'' - c_1'} = \mathrm{I}\,\frac{d_2'' - d_1''}{d_1'' - c_1'} + \mathrm{I}\,\frac{c_1' - c_2'}{d_1'' - c_1'} \ldots (200).$$

The sought axis of the cone of revolution through the sides cc_1c_2 of the cyclic cone, or a line in the direction of this axis, is therefore thus given, by the expression (200), as the symbolic *sum* of two other lines; which two new lines, by comparison of their expressions with the form (188), are seen to be in the directions of the axes of revolution of two new or *auxiliary cones* of revolution; one of these auxiliary cones containing, upon a single sheet, the three lines

$$c_1',\ d_1'',\ d_2'' \ldots\ldots\ldots\ldots\ldots (201),$$

so that it may be briefly called the cone of revolution $c_1'\ d_1''\ d_2''$; while the other auxiliary cone of revolution, which may be called in like manner the cone $c_2'\ c_1'\ d_1''$, contains on one sheet this other system of three straight lines,

$$c_1',\ d_1'',\ c_2' \ldots\ldots\ldots\ldots\ldots (202).$$

The symbolic *difference* of the same two lines, namely, that of the lines denoted by the symbols

$$\mathrm{I}\,\frac{d''_2 - d''_1}{d''_1 - c'_1},\quad \mathrm{I}\,\frac{c'_1 - c'_2}{d''_1 - c'_1} \ldots\ldots\ldots\ldots (203),$$

which lines are thus in the directions of the axes of these two new cones of revolution, may easily be expressed under the form

$$\mathrm{I}\,\frac{\frac{1}{2}(d''_2 + c_2') - \frac{1}{2}(d_1'' + c_1')}{\frac{1}{2}(d_1'' + c_1') - c_1'} \ldots\ldots\ldots (204);$$

it is therefore (by the same last article) a line perpendicular to the plane in which the three following straight lines terminate, if drawn from one common point, such as the common vertex of the four cones,

$$c_1',\quad \tfrac{1}{2}(d_1'' + c_1'),\quad \tfrac{1}{2}(d_2'' + c_2') \ldots\ldots (205).$$

This plane contains also the termination of the line d_1'', if that line be still drawn from the same vertex; because, in general, whatever may be the value of the scalar x, the three straight lines denoted by the symbols

$$c_1', \quad (1 - x)\, d_1'' + x c_1', \quad d_1'' \ldots\ldots\ldots (206),$$

all terminate on one straight line, if they be drawn from one common origin; and this last straight line is situated in the first secant plane, and connects the extremities of the two equally long lines c_1', d_1'', which are the traces of that secant plane on the two cyclic planes. The remaining line, $\frac{1}{2}(d_2'' + c_2')$, of the system (205), if still drawn from the same vertex as before, bisects that other straight line, situated in the second secant plane, which connects the extremities of the two equally long traces c_2', d_2'', of that other secant plane on the same two cyclic planes. And these two connecting lines, thus situated respectively in the first and second secant planes do not generally intersect each other; because they cut the line of mutual intersection of those two secant planes, namely the side c of the given cyclic cone, in points which are in general situated at different distances from the vertex. It is therefore in general a determinate problem, to draw through the first of these two connecting lines a plane which shall bisect the second: and we see that the plane so drawn, being that in which the three lines (205) terminate, is perpendicular to the line (204), that is to the symbolic difference,

$$\mathrm{I}\,\frac{d_2'' - d_1''}{d_1'' - c_1'} - \mathrm{I}\,\frac{c_1' - c_2'}{d_1'' - c_1'} \ldots\ldots\ldots\ldots (207),$$

of the two lines (203), of which the symbolic sum (200) has been seen to be a line in the direction of the axis (197) of the first cone of revolution considered in the present article; while the two lines (203), of which we have thus taken the symbolic sum and difference, have been perceived to be in the directions of the axes of the two other and auxiliary cones of revolution, which we have also had occasion to consider. But in general, by one of those fundamental principles which the present system of symbolical geometry has in *common* with other systems, the symbolical sum and difference of two adjacent and coinitial sides of a parallelogram may be represented or constructed geometrically by the two diagonals of that figure; namely the sum by that diagonal which is intermediate between the two sides, and the difference by that other diagonal which is transversal to those sides: and

every other transversal straight line, which is drawn across the same two sides in the same direction as the second diagonal, is bisected by the first diagonal, because the two diagonals themselves bisect each other. We may therefore enunciate this theorem:—*If across the axes* (203) *of the two auxiliary cones of revolution, which contain respectively the two systems of straight lines* (201) *and* (202), (each system of three straight lines being contained upon a single sheet), *we draw a rectilinear transversal, perpendicular to the plane which contains the first and bisects the second of the two connecting lines, drawn as before in the two secant planes; and if we then bisect this transversal by a straight line drawn from the common vertex of the cones: this bisecting line will be situated on the axis of revolution* (197) *of that other cone of revolution, which contains upon one sheet the three given sides of the given cyclic cone.* (The drawing of this transversal is possible, because the preceding investigation shews that the plane of the axes of revolution of the two auxiliary cones is perpendicular to that other plane which is described in the construction.)

28. Since, generally, in the present system of symbolical geometry, the vector part of the quotient of any two parallel lines, and the scalar part of the quotient of any two perpendicular lines, are respectively equal to zero, we may express that *three* straight lines, a, b, c, if drawn from a common origin, all *terminate on one common straight line*, by writing the equation

$$\mathrm{V}\,\frac{c-a}{b-a}=0 \;\ldots\ldots\ldots\; (208);$$

and may express that *two* straight lines, a, c, are *equally long*, or that they are fit to be made adjacent sides of a rhombus (of which the two diagonals are mutually rectangular), by this other formula:

$$\mathrm{S}\,\frac{c+a}{c-a} \;\ldots\ldots\ldots\ldots\ldots\; (209).$$

If then we combine these two conditions, which will give

$$\mathrm{S}\,\frac{c+a}{b-a}=0 \;\ldots\ldots\ldots\ldots\; (210),$$

and therefore

$$\mathrm{S}\,\frac{c}{b-a}=-\,\mathrm{S}\,\frac{a}{b-a},\quad \mathrm{V}\,\frac{c}{b-a}=\mathrm{V}\,\frac{a}{b-a} \;\ldots\; (211),$$

we shall thereby express that the chord or secant of a circle or sphere, which passes through the extremity of one given

radius a, and also through the extremity of another given and coinitial straight line b, meets the circumference of the same circle or the surface of the same sphere again at the extremity of the other straight line denoted by c, which will thus be another radius. But with the same mode of abridgment as that employed in the formula (178), we have, by (211),

$$(V + S)\frac{c}{b - a} = (V - S)\frac{a}{b - a} \;\ldots\ldots\; (212),$$

and therefore

$$c = (V - S)\frac{a}{b - a} \,.\, (b - a) \;\ldots\ldots\ldots\; (213).$$

This last is consequently an expression for the second radius c, in terms of the first radius a, and of the other given line b from the same centre, which terminates at some given point upon the common chord or secant, connecting the extremities of the two radii. If therefore we write for abridgment

$$m = \tfrac{1}{2}(d_2'' + c_2') \ldots\ldots\ldots\ldots\ldots (214),$$

so that m shall be a symbol for the last of the three lines (205); and if we employ the two following expressions, formed on the plan (213),

$$\left.\begin{aligned} m' &= (V - S)\frac{c_1'}{m - c_1'} \,.\, (m - c_1') \\ m'' &= (V - S)\frac{d_1''}{m - d_1''} \,.\, (m - d_1'') \end{aligned}\right\} \ldots\ldots (215),$$

the symbols c_1', d_1'' retaining their recent meanings; then the four straight lines,

$$c_1',\; d_1'',\; m',\; m'' \ldots\ldots\ldots\ldots\ldots (216),$$

all drawn from the given vertex of the cones, will be equally long, and will terminate in four concircular points; or, in other words, their extremities will be the four corners of a certain quadrilateral inscribed in a circle: of which plane quadrilateral the two diagonals, connecting respectively the ends of c_1', m', and of d_1'', m'', will intersect each other at the extremity of the line m, which is drawn from the same vertex as before. It may also be observed respecting this line m, that in virtue of its definition (214), and of the second parallelism (193), it bisects the angle between the two equally long sides c, c_2, of the given cyclic cone. Thus *the four lines* (216) *are four sides of one common sheet of a new cone of revolution, of which the axis is perpendicular to the plane described in the construction of the foregoing article;*

because these four equally long lines (216) terminate on the same plane as the three lines (205), that is on a plane perpendicular to the line (204) or (207), which latter line has thus the direction of the axis of revolution of the new auxiliary cone. It is usual to say that four diverging straight lines are *rays of an harmonic pencil*, or simply that they are *harmonicals*, when a rectilinear transversal, parallel to the fourth, and bounded by the first and third, is bisected by the second of these lines: so that, in general, any four diverging straight lines which can be represented by the four symbols

$$a, \; a + b, \; b, \; a - b,$$

or by symbols which are obtained from these by giving them any scalar coefficients, have the *directions* of four such harmonicals. We are then entitled to assert that *the fourth harmonical to the axes of the three cones of revolution*

$$(c_1'd_1''d_2''), \; (cc_1c_2), \; (c_2'c_1'd_1'') \ldots\ldots\ldots(217),$$

which three axes have been already seen to be all situated in one common plane, *is the axis of that new or fourth cone of revolution* $(c_1'd_1''m'm'')$, *which contains on one sheet the four straight lines* (216). And if we regard the four last-mentioned lines as *edges of a tetrahedral angle*, inscribed in this new cone of revolution, we see that *the two diagonal planes* of this tetrahedral angle *intersect each other along a straight line* m, *which bisects the plane angle* (c, c_2) *between two of the edges of the trihedral angle* (cc_1c_2); which latter angle is at once inscribed in the given cyclic cone, and also in that cone of revolution which it was originally proposed to construct.

29. Conceive now that this original cone of revolution (cc_1c_2) comes to *touch* the given cyclic cone along the side c, as a consequence of a gradual and unlimited approach of the second secant plane (cc_2), to coincidence with the given tangent plane $(c'cd')$, which touches the given cone along that side; or in virtue of a gradual and indefinite tendency of the side c_2 to coincide with the given side c. The line m, bisecting always the angle between these two sides c, c_2, will thus itself also tend to coincide with c; and the diagonal planes of the tetrahedral angle $(c_1'd_1''m'm'')$, which planes still intersect each other in m, will tend at the same time to contain the same given side. But that side c is (by the construction) a line in the plane of one face of that tetrahedral angle, namely in the plane of c_1' and d_1'', which was the first secant plane of the cyclic cone; consequently the tetrahedral angle itself, and its circumscribed

cone of revolution, tend generally to flatten together into coincidence with this secant plane, as c_2 thus approaches to c: and the axis of the cone ($c_1'd_1''m'm''$) coincides ultimately with the normal to the first secant plane ($c_1'd_1''$). At the same time the traces c_2' and d_2'', of the second secant plane on the two cyclic planes, tend to coincide with the traces c' and d'' of the given tangent plane thereupon. We have therefore this new theorem, which is however only a limiting form of that enunciated in article 27:—If through a given side (c) of a given cyclic cone, we draw a tangent plane ($c'cd''$), and a secant plane ($c_1'cc_1d_1''$); and if we then describe three cones of revolution, the first of these three cones containing on one sheet the two traces (c_1', d_1'') of the secant plane, and one trace (d'') of the tangent plane; the second cone of revolution touching the cyclic cone along the side of contact (c), and cutting it along the side of section (c_1); and the third cone of revolution containing the same two traces (c_1', d_1'') of the secant plane, and the other trace (c') of the tangent plane: *the fourth harmonical to the axes of revolution of these three cones will be perpendicular to the secant plane.*

30. Finally, conceive that the remaining secant plane $c_1'd_1''$ tends likewise to coincide with the tangent plane $c'd''$; the cone of revolution which lately *touched* the given cyclic cone along the given side c, will now come to *osculate* to that cone along that side: and because a line in the direction of the mutual intersection of the two cyclic planes has been already denoted by a', therefore the first and third of the three last-mentioned cones of revolution tend now to touch the planes $a'd''$ and $a'c'$, respectively, along the lines d'' and c'. The theorem of article 27, at the limit here considered, takes therefore this new form:—*If three cones of revolution be described, the first cone cutting the first cyclic plane* ($a'c'$) *along the first trace* (c') *of a given tangent plane* ($c'cd''$) *to a given cyclic cone, and touching the second cyclic plane* ($a'd''$) *along the second trace* (d'') *of the same tangent plane; the second cone of revolution osculating to the same cyclic cone, along the given side of contact* (c); *and the third cone of revolution touching the first cyclic plane and cutting the second cyclic plane, along the same two traces as before: then the fourth harmonical to the axes of revolution of these three cones will be the normal to the plane* ($c'd''$) *which touches at once the given cyclic cone, and the sought osculating cone, along the side* (c) *of contact or of osculation.*

31. To deduce from this last theorem an *expression* for a line e in the direction of the axis of the osculating cone of revolution, by the processes of this symbolical geometry, we may remark in the first place, that when any two straight lines a, b, are equally long, we have the three equations following:

$$\mathrm{S}\frac{\mathrm{a}}{\mathrm{b}} = \mathrm{S}\frac{\mathrm{b}}{\mathrm{a}}, \quad \mathrm{V}\frac{\mathrm{a}}{\mathrm{b}} = -\mathrm{V}\frac{\mathrm{b}}{\mathrm{a}}, \quad \mathrm{I}\frac{\mathrm{a}}{\mathrm{b}} = -\mathrm{I}\frac{\mathrm{b}}{\mathrm{a}} \ldots\ldots (218),$$

from the two former of which it may be inferred that the relation

$$\frac{\mathrm{V}}{\mathrm{S}} \cdot \frac{\mathrm{a}}{\mathrm{b}} = -\frac{\mathrm{V}}{\mathrm{S}} \cdot \frac{\mathrm{b}}{\mathrm{a}} \ldots\ldots\ldots\ldots (219)$$

holds good, not only when the two lines a, b are thus equal in length, but generally for any two lines: because if we multiply or divide either of them by any scalar coefficient, we only change thereby in one common (scalar) ratio both the scalar and vector parts of their quotient, and so do not affect that other quotient which is obtained by dividing the latter of these two parts by the former. We may also obtain the equation (219), as one which holds good for any two straight lines a, b, under the form

$$\mathrm{S}\frac{\mathrm{b}}{\mathrm{a}}\mathrm{V}\frac{\mathrm{a}}{\mathrm{b}} + \mathrm{V}\frac{\mathrm{b}}{\mathrm{a}}\mathrm{S}\frac{\mathrm{a}}{\mathrm{b}} = 0 \ldots\ldots\ldots\ldots (220),$$

by operating with the characteristic V on the identity,

$$\mathrm{S}\frac{\mathrm{b}}{\mathrm{a}} \cdot \frac{\mathrm{a}}{\mathrm{b}} + \mathrm{V}\frac{\mathrm{b}}{\mathrm{a}} \cdot \frac{\mathrm{a}}{\mathrm{b}} = \frac{\mathrm{b}}{\mathrm{b}} = 1 \ldots\ldots\ldots (221);$$

while, if we operate on the same identity (221) by the characteristic S, we obtain this other general formula, which likewise holds good for any two straight lines a, b, whether equal or unequal in length, and will be useful to us on future occasions,

$$\mathrm{S}\frac{\mathrm{b}}{\mathrm{a}}\mathrm{S}\frac{\mathrm{a}}{\mathrm{b}} + \mathrm{V}\frac{\mathrm{b}}{\mathrm{a}}\mathrm{V}\frac{\mathrm{a}}{\mathrm{b}} = 1 \ldots\ldots\ldots (222).$$

Again, if there be three equally long lines, a, b, c, then since the principle contained in the third equation (218) gives

$$\mathrm{I}\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}} = \mathrm{I}\frac{\mathrm{b}}{\mathrm{c}} - \mathrm{I}\frac{\mathrm{a}}{\mathrm{c}} = \mathrm{I}\frac{\mathrm{c}}{\mathrm{a}} - \mathrm{I}\frac{\mathrm{c}}{\mathrm{b}} \ldots\ldots (223),$$

which last expression is only multiplied by a scalar when the line c is multiplied thereby; while the index of a geometrical fraction is (among other properties) a line perpendicular to

both the numerator and denominator of the fraction; we see that the symbol $\mathrm{I}\,\frac{c}{a} - \mathrm{I}\,\frac{c}{b}$ denotes generally a line perpendicular to both c and b − a, if only the two lines a and b have their own lengths equal to each other, without any restriction being thereby laid on the length of c: this symbol denotes therefore, under this single condition, a straight line contained in a plane perpendicular to c, and having equal inclinations to a and b. Thus, under the same condition, the same symbol $\mathrm{I}\,\frac{c}{a} - \mathrm{I}\,\frac{c}{b}$ may represent the axis d of a cone of revolution, which contains upon one sheet the two equally long lines a and b, while the third line c is in or parallel to the *single* cyclic plane of this *monocyclic cone*, or the plane of its circular base, or of one of its circular sections; or coincides with or is parallel to some tangent to such circular base or section. If then we know any other line a′, contained in the plane which touches this monocyclic cone along the side a, we may substitute for c, in this symbol $\mathrm{I}\,\frac{c}{a} - \mathrm{I}\,\frac{c}{b}$, that part or component of this new line a′ which is perpendicular to the side of contact a; and therefore may write with this view,

$$c = \mathrm{V}\,\frac{a'}{a}\,.\,a = a' - \mathrm{S}\,\frac{a'}{a}\,.\,a \; \ldots\ldots\ldots \; (224),$$

which will give $$d = \mathrm{I}\,\frac{a'}{a} - \mathrm{I}\,\frac{a'}{b} + \mathrm{S}\,\frac{a'}{a}\,\mathrm{I}\,\frac{a}{b} \; \ldots\ldots\ldots \; (225),$$

as a general expression for a line d in the direction of the axis of a cone of revolution which is touched by the plane aa′ along the side of contact a, and contains on the same sheet the equally long side b. We may also remark that because the normal plane to a cone of revolution, drawn along any side of that cone, contains the axis of revolution, so that the plane containing the axis and the side is perpendicular to the tangent plane, we have a relation between the three directions of a, a′, d, which does not involve the direction of b, and may be expressed by any one of the three following formulæ:—

$$\angle\,(a', a, d) = \frac{\pi}{2}, \quad d \perp \mathrm{V}\,\frac{a'}{a}\,.\,a, \quad \mathrm{S}\,\frac{a'}{d} = \mathrm{S}\,\frac{a'}{a}\,\mathrm{S}\,\frac{a}{d} \ldots . \; (226);$$

in each of which it is allowed to reverse the direction of d, or to change d to − d. (Compare the formulæ (168), for the

notation of dihedral angles.) It may indeed be easily proved, without the consideration of any cone, that any one of these three formulæ (226) involves the other two; but we see also by the recent reasoning, that they may all be deduced when an expression of the form (225) for d is given; or when this line d can be expressed in terms of a, a′, and of another line b which is supposed to have the same length as a, by any symbol which differs only from the form (225) through the introduction of a scalar coefficient.

These things being premised, if we change a, b, d, in this form (225), to c′, d″, n′, we find

$$n' = \mathrm{I}\frac{a'}{c'} - \mathrm{I}\frac{a'}{d''} + \mathrm{S}\frac{a'}{c'}\,\mathrm{I}\frac{c'}{d''} \;\ldots\ldots\; (227),$$

as an expression for a line n′ in the direction of the axis of revolution of the cone which touches the first cyclic plane a′c′ along the first trace c′ of the tangent plane, and cuts the second cyclic plane a′d″ along the second trace d′ of the same tangent plane; that is to say, in the direction of the axis of the third cone of revolution, described in the enunciation of the theorem of article 30. Again, if we change a, b, d, in the same general formula (225), to d″, c′, − n″, and attend to the third equation (218), we find

$$n'' = \mathrm{I}\frac{a'}{c'} - \mathrm{I}\frac{a'}{d''} + \mathrm{S}\frac{a'}{d''}\,\mathrm{I}\frac{c'}{d''} \ldots\ldots\ldots (228),$$

as an expression for another line n″, in the direction of the axis of another cone of revolution, which cuts the first cyclic plane a′c′ along the trace c′, and touches the second cyclic plane a′d″ along the other trace d″ of the tangent plane; that is, in the direction of the axis of revolution of the first of the three cones, described in the enunciation of the same theorem of article 30. And since these expressions give

$$n'' - n' = \left(\mathrm{S}\frac{a'}{d''} - \mathrm{S}\frac{a'}{c'}\right)\mathrm{I}\frac{c'}{d''} \;\ldots\ldots\; (229),$$

we have the two perpendicularities

$$n'' - n' \perp c', \quad n'' - n' \perp d'' \ldots\ldots\ldots (230);$$

so that a transversal drawn across the two axes of revolution last determined, in the direction of this symbolic difference n″ − n′, is perpendicular to both the traces of the tangent plane c′d″, and therefore has the direction of the normal to that plane, or to the cyclic cone; or, in other words, this transversal has the direction of the fourth harmonical mentioned in the theorem. But the lines n″ and n′, of which

the symbolic *difference* has thus been taken, have been seen to be in the directions of the first and third of the same four harmonicals; and the axis of the osculating cone, which axis we have denoted by e, has (by the theorem) the direction of the second harmonical: it has therefore the direction of the symbolical *sum* of the same two lines n'', n', because it bisects their transversal drawn as above. Thus by conceiving the bisector to terminate on the transversal, we find, as an expression for this sought axis e, the following,

$$e = \tfrac{1}{2}(n'' + n') = I\,\frac{a'}{c'} - I\,\frac{a'}{d''} + \tfrac{1}{2}\left(S\,\frac{a'}{c'} + S\,\frac{a'}{d''}\right) I\,\frac{c'}{d''} \quad ..(231).$$

32. This symbolical expression for e contains, under a not very complex form, the solution of the problem on which we have been engaged; namely, *to find the axis of the cone of revolution, which osculates along a given side to a given cyclic cone.* It may however be a little simplified, and its geometrical interpretation made easier, by resolving the line a' into two others, which shall be respectively parallel and perpendicular to the *lateral normal plane,* as follows:

$$a' = a^{\backprime} + a^{\backprime\backprime}; \quad a^{\backprime} \perp d'' - c'; \quad a^{\backprime\backprime} \parallel d'' - c' \quad ..(232);$$

so that

$$a^{\backprime} = V\,\frac{a'}{d'' - c'}\,.(d'' - c'); \quad a^{\backprime} = S\,\frac{a'}{d'' - c'}\,.(d'' - c') \quad ..(233);$$

which will give, by (191) and (218), because $a^{\backprime\backprime} \perp d'' + c'$,

$$S\,\frac{d''}{a^{\backprime\backprime}} + S\,\frac{c'}{a^{\backprime\backprime}} = 0; \quad S\,\frac{a^{\backprime\backprime}}{c'} + S\,\frac{a^{\backprime\backprime}}{d''} = 0 \quad ..\ (234);$$

also

$$I\,\frac{d''}{a^{\backprime\backprime}} - I\,\frac{c'}{a^{\backprime\backprime}} = 0; \quad I\,\frac{a^{\backprime\backprime}}{c'} - I\,\frac{a^{\backprime\backprime}}{d''} = 0 \quad ..\ (235);$$

and

$$S\,\frac{d''}{a^{\backprime}} - S\,\frac{c'}{a^{\backprime}} = 0; \quad S\,\frac{a^{\backprime}}{c'} - S\,\frac{a^{\backprime}}{d''} = 0 \quad ..\ (236).$$

For by thus resolving a', in (231), into the two components $a^{\backprime}$ and $a^{\backprime\backprime}$, it is at once seen, by (234) (235), that the latter component $a^{\backprime\backprime}$ disappears from the result, which reduces itself by (236) to the following simplified form,

$$e = I\,\frac{a^{\backprime}}{c'} - I\,\frac{a^{\backprime}}{d''} + S\,\frac{a^{\backprime}}{c'}\,I\,\frac{c'}{d''} \quad \ldots\ldots\ (237);$$

and this gives, by comparison with the forms (225) and (226), a remarkable relation of rectangularity between two planes,

of which one contains the axis e of the osculating cone, namely the planes $a^{\backprime}c'$ and $c'e$; which relation is expressed by the formula,

$$\angle\,(a^{\backprime},\, c',\, e) = \frac{\pi}{2} \ldots\ldots\ldots\ldots (238).$$

In like manner, from the same expression (231), by the same decomposition of a', we may easily deduce, instead of (237), this other expression for the axis of the osculating cone,

$$e = \mathrm{I}\,\frac{a^{\backprime}}{c'} - \mathrm{I}\,\frac{a^{\backprime}}{d''} - \mathrm{S}\,\frac{a^{\backprime}}{d''}\,\mathrm{I}\,\frac{d''}{c'} \ldots\ldots (239);$$

and may derive from it this other relation, of rectangularity between two other planes, namely the planes $a^{\backprime}d''$ and $d''e$,

$$\angle\,(a^{\backprime},\, d'',\, e) = \frac{\pi}{2}, \ldots\ldots\ldots\ldots (240).$$

Hence follows immediately this theorem, which furnishes a remarkably simple *construction with planes*, for determining generally a line in the required direction of the axis of the osculating cone:—*If we project the line* a' *of mutual intersection of the two cyclic planes* $a'c'$, $a'd''$, *of any given cyclic cone, on the lateral normal plane which is drawn along any given side* c; *if we next draw two planes*, $a^{\backprime}c'$, $a^{\backprime}d''$, *through the projection* $a^{\backprime}$ *thus obtained, and through the two traces*, c', d'', *of the tangent plane on the two cyclic planes; and if we then draw two new planes*, $c'e$, $d''e$, *through the same two traces of the tangent plane, perpendicular respectively to the two planes* $a^{\backprime}c'$, $a^{\backprime}d''$, *last drawn: these two new planes will intersect each other along the axis* e *of the cone of revolution, which osculates along the given side* c *to the given cyclic cone.*

And by considering, instead of these cones and planes, their intersections with a spheric surface described about the common vertex, we arrive at the following *spherographic construction*,* for finding the *spherical centre of curvature*

* This construction was communicated to the Royal Irish Academy (see *Proceedings*), at its meeting of November 30th, 1847, along with a simple geometrical construction for generating a system of two reciprocal ellipsoids by means of a moving sphere, as new applications of the author's Calculus of Quaternions to Surfaces of the Second Order. With that Calculus, of which the fundamental principles and formulæ were communicated to the same Academy on the 13th of November, 1843, it will be found that the present System of Symbolical Geometry is connected by very intimate relations, although the subject is approached, in the two methods, from two quite different points of view: the *algebraical quaternion* of the one method being *ultimately* the same as the *geometrical fraction* of the other.

of a given spherical conic at a given point, or the pole of the small circle which osculates at that point to that conic:—*From one of the two points of mutual intersection of the two cyclic arcs let fall a perpendicular upon the normal arc to the conic, which latter arc is drawn through the given point of osculation; connect the foot of this (arcual) perpendicular by two other arcs of great circles, with those two known points, equidistant from the point upon the conic, where the tangent arc meets the two cyclic arcs; draw through the same two points two new arcs of great circles, perpendicular respectively to the two connecting arcs: these two new arcs will cross each other on the normal arc, in the pole of the osculating circle, or in the spherical centre of curvature of the spherical conic,* which centre it was required to determine.

[*To be continued.*]

THEOREMS WITH REFERENCE TO THE SOLUTION OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS.

By William Thomson.

Theorem 1. It is possible to find a function V, of x, y, z,* which shall satisfy, for all real values of these variables, the differential equation

$$\frac{d\left(a^2 \frac{dV}{dx}\right)}{dx} + \frac{d\left(a^2 \frac{dV}{dy}\right)}{dy} + \frac{d\left(a^2 \frac{dV}{dz}\right)}{dz} = -4\pi\rho \ldots \text{(A)},$$

a being any real continuous or discontinuous function of x, y, z, and ρ a function which vanishes for all values of x, y, z, exceeding certain finite limits (such as may be represented geometrically by a finite closed surface), within which its value is finite, but entirely arbitrary.

Theorem 2. There cannot be two different solutions of equation (A) for all real values of the variables.

1. (*Demonstration*). Let U be a function of x, y, z, given by the equation

$$U = \iiint \frac{\rho' dx' dy' dz'}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{1}{2}}} \ldots\ldots (a),$$

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* The case of three variables, which includes the applications to physical problems, is alone considered here; although the analysis is equally applicable whatever be the number of variables.

the integrations in the second member including all the space for which ρ' is finite; so that, if we please, we may conceive the limits of each integration to be $-\infty$ and $+\infty$, as thus all the values of the variables for which ρ' is finite will be included, and the amount of the integral will not be affected by those values of the variables for which ρ' vanishes, being included. Again, V being any real function of x, y, z, let

$$Q=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\left(a\frac{dV}{dx}-\frac{1}{a}\frac{dU}{dx}\right)^2+\left(a\frac{dV}{dy}-\frac{1}{a}\frac{dU}{dy}\right)^2\right.$$
$$\left.+\left(a\frac{dV}{dz}-\frac{1}{a}\frac{dU}{dz}\right)^2\right\}dx\,dy\,dz \;\ldots\ldots\; (b).$$

It is obvious that, although V may be assigned so as to make Q as great as we please, it is impossible to make the value of Q less than a certain limit, since we see at once that it cannot be negative. Hence Q, considered as depending on the arbitrary function V, is susceptible of a minimum value; and the calculus of variations will lead us to the assigning of V according to this condition.

Thus we have

$$-\tfrac{1}{2}\delta Q=\iiint\left\{\left(a\frac{dV}{dx}-\frac{1}{a}\frac{dU}{dx}\right).\,a\frac{d\delta V}{dx}+\left(a\frac{dV}{dy}-\frac{1}{a}\frac{dU}{dy}\right).\,a\frac{d\delta V}{dy}\right.$$
$$\left.+\left(a\frac{dV}{dz}-\frac{1}{a}\frac{dU}{dz}\right).\,a\frac{d\delta V}{dz}\right\}dx\,dy\,dz.$$

Hence, by the ordinary process of integration by parts, the integrated terms vanishing at each limit,* we deduce

$$-\tfrac{1}{2}\delta Q=\iiint\delta V\left\{\frac{d}{dx}\left(a^2\frac{dV}{dx}-\frac{dU}{dx}\right)+\frac{d}{dy}\left(a^2\frac{dV}{dy}-\frac{dU}{dy}\right)\right.$$
$$\left.+\frac{d}{dz}\left(a^2\frac{dV}{dz}-\frac{dU}{dz}\right)\right\}dx\,dy\,dz.$$

But by a well-known theorem (proved in Pratt's *Mechanics*, and in the treatise on Attraction in Earnshaw's *Dynamics*), we have

$$\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}+\frac{d^2U}{dz^2}=-4\pi\rho.$$

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* All the functions of x, y, z contemplated in this paper are supposed to vanish for infinite values of the variables.

Hence the preceding expression becomes

$$-\tfrac{1}{2}\delta Q = \iiint \delta V . \left\{\frac{d}{dx}\left(a^2 \frac{dV}{dx}\right) + \frac{d}{dy}\left(a^2 \frac{dV}{dy}\right) + \frac{d}{dz}\left(a^2 \frac{dV}{dz}\right) + 4\pi\rho\right\} dx\, dy\, dz.$$

We have therefore, for the condition that Q may be a maximum or minimum, the equation,

$$\frac{d}{dx}\left(a^2 \frac{dV}{dx}\right) + \frac{d}{dy}\left(a^2 \frac{dV}{dy}\right) + \frac{d}{dz}\left(a^2 \frac{dV}{dz}\right) = -4\pi\rho,$$

to be satisfied for all values of the variables.

Now it is possible to assign V so that Q may be a minimum, and therefore there exists a function, V, which satisfies equation (A).

2. (*Demonstration*). Let V be a solution of (A), and let V_1 be any different function of x, y, z, that is to say, any function such that $V_1 - V$, which we may denote by ϕ, does not vanish for all values of x, y, z. Let us consider the integral Q_1, obtained by substituting V_1 for V, in the expression for Q. Since

$$\left(a \frac{dV_1}{dx} - \frac{1}{a}\frac{dU}{dx}\right)^2$$
$$= \left(a \frac{dV}{dx} - \frac{1}{a}\frac{dU}{dx}\right)^2 + 2\left(a \frac{dV}{dx} - \frac{1}{a}\frac{dU}{dx}\right) a \frac{d\phi}{dx} + a^2 \frac{d\phi^2}{dx^2},$$

we have

$$Q_1 = Q + 2\iiint\left\{\left(a \frac{dV}{dx} - \frac{1}{a}\frac{dU}{dx}\right) a \frac{d\phi}{dx} + \left(a \frac{dV}{dy} - \frac{1}{a}\frac{dU}{dy}\right) a \frac{d\phi}{dy} + \left(a \frac{dV}{dz} - \frac{1}{a}\frac{d\phi}{dz}\right) a \frac{d\phi}{dz}\right\} dx\, dy\, dz$$
$$+ \iiint a^2 \left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right) dx\, dy\, dz.$$

Now, by integration by parts, we find

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(a \frac{dV}{dx} - \frac{1}{a}\frac{dU}{dx}\right) a \frac{d\phi}{dx} . dx\, dy\, dz$$
$$= -\iiint \phi . \frac{d}{dx}\left(a^2 \frac{dV}{dx} - \frac{dU}{dx}\right) dx\, dy\, dz,$$

the integrated term vanishing at each limit. Applying this and similar processes with reference to y and z, we find an

expression for the second term of Q_1 which, on account of equation (A) vanishes. Hence

$$Q_1 = Q + \iiint a^2 \left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz}\right) dx\,dy\,dz \ldots\ldots(c),$$

which shews that Q_1 is greater than Q. Now the only peculiarity of Q is, that V, from which it is obtained, satisfies the equation (A), and therefore V_1 cannot be a solution of (A). Hence no function different from V can be a solution of (A).

The analysis given above, especially when interpreted in various cases of abrupt variations in the value of a, and of infinite or evanescent values, through finite spaces, possesses very important applications in the theories of heat, electricity, magnetism, and hydrodynamics, which may form the subject of future communications.

Edenbarnet, Dumbartonshire, Oct. 9, 1847.

NOTE ON THE INTEGRATION OF THE EQUATIONS OF EQUILIBRIUM OF AN ELASTIC SOLID.

By William Thomson.

In a short paper published at the beginning of this year (1847; see *Mathematical Journal*, vol. II. p. 61) certain particular integrals of the equations of equilibrium of an incompressible elastic solid were given. It may readily be shewn that any integral whatever may be expressed by the sum of such particular integrals as those which are given in equations (4) of the paper referred to; or that any possible state of distortion of an incompressible solid may be considered as the resultant of coexistent elementary states of distortion, each of which is represented by a particular solution of the form there expressed. Hence the consideration of the particular integrals alluded to may be of great importance in the mathematical theory of incompressible solids; and it would be very desirable to have a similar method for treating the theory of elastic solids in general. I have recently found that a slight modification may be introduced in the expressions already referred to, by which their application will be extended from the case of incompressible solids to that of solids in general. Thus, l, m, n, and f, g being arbitrary constants, if we take the expressions written below, for α, β, γ, the three components of the displacement of a point (x, y, z) from its position of equilibrium, we have, by means

of the following assumptions, a solution of the general equations of equilibrium, for a solid of any kind, which will become the same as equations (4) of the former paper, corresponding to the case of an incompressible solid, if we put $f = g$.

Let
$$\alpha = \tfrac{1}{2} f \frac{d}{dx} \frac{lx + my + nz}{r} - g\frac{l}{r},$$
$$\beta = \tfrac{1}{2} f \frac{d}{dy} \frac{lx + my + nz}{r} - g\frac{m}{r},$$
$$\gamma = \tfrac{1}{2} f \frac{d}{dz} \frac{lx + my + nz}{r} - g\frac{n}{r},$$
where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$.

To shew that these equations satisfy the three general equations of equilibrium, of which the following is one, and the other two complete a symmetrical system,*

$$k\frac{d}{dx}\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) + \frac{d^2\alpha}{dx^2} + \frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2} = 0,$$

we have merely to perform the necessary differentiations and substitutions. Thus we find

$$\frac{d^2\alpha}{dx^2} + \frac{d^2\alpha}{dy^2} + \frac{d^2\alpha}{dz^2} = -f\frac{d}{dx}\frac{lx + my + nz}{r^3},$$
$$\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = (-f + g)\frac{lx + my + nz}{r^3}.$$

Hence, by substitution in the first member of the preceding equation, we find that it vanishes if

$$k(g - f) - f = 0.$$

This equation may be put under the form

$$\frac{f}{g} = \frac{k}{k+1}.$$

If now we suppose the solid to be incompressible, we must take $k = \infty$. This gives $f = g$, and we have the case of the former paper.

The application of the solution expressed above, in the mathematical theory of elastic solids, is analogous in some

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* Already referred to in the *Math. Journal*, Second Series, Vol. II. p. 62. See Stokes on Elastic Solids, *Camb. Trans.* Vol. VIII. Part III., the only work in which the true formulæ are given.

degree to a method of treating certain questions in the theory of heat, which was indicated in a paper "On the Uniform Motion of Heat in Solid Bodies, and its Connexion with the Mathematical Theory of Electricity." Instead of the three components of the flux of heat at any point, we have to consider the three components of the rotation of an element of a distorted solid; instead of a source of heat, we have a *source of strain* round the point of application of a force. If the solid were incompressible, there would be as close a connexion with the mathematical theory of electro-magnetism, as was shewn to subsist between the theories of heat and electricity: this follows at once from the theorem given in the former paper, with reference to the "mechanical representation" of the electro-magnetic force due to an element of a galvanic arc.

I cannot at present enter upon the general mathematical theory of elastic solids; but the method here indicated may be readily imagined by examining the meaning of the particular solution given above. It is easily shewn that those forms for α, β, γ express the three components of the displacement produced at a point (x, y, z) in an infinite homogeneous elastic solid, where a force is applied to it at the origin of coordinates, in the direction (l, m, n).

Glasgow College, December 10, 1847.

NOTES ON HYDRODYNAMICS.

II.—*On the Equation of the Bounding Surface.*

By William Thomson.

If a fluid mass be in motion, under any conceivable circumstances, its bounding surface will always be such that there will be no motion of the fluid across it. By expressing this circumstance analytically, we obtain the differential equation of the bounding surface.*

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* The condition to be expressed is entirely equivalent to this: that the surface must always contain the same fluid matter within it, and it therefore appears that the differential equation expressing this condition, according to the investigation in the text, is satisfied, without exception, at the extreme boundary of the fluid, as well as at the varying surface bounding any portion of the entire mass. Mr. Stokes is, I believe, the only writer who has given this view of the subject, all other authors having taken for the condition to be expressed, that a particle which is once in the bounding surface always remains in the bounding surface. Poisson has justly re-

In most problems of Hydrodynamics certain conditions are given with reference to the surface within which the fluid mass considered is contained: for instance, that this surface is given as fixed in form and position; that its form or position varies in a given arbitrary manner; or, as may be the case when the mass considered is a *liquid*, that the bounding surface is free, but subjected to a given constant pressure. Hence, in the analysis of all such problems it will be necessary to have an expression of the fact that the fluid is actually contained within the surface at which the given conditions are satisfied. This will be effected by combining the differential equation of the surface with the special equation, or equations, of condition in each particular problem, all of which must be satisfied at the bounding surface of the fluid. Hence the investigation of the equation of the surface ought to find a place in a complete treatise on the mathematical theory of hydrodynamics. It is wanting in many of the elementary works, and this paper may therefore be considered to be useful as supplying the deficiency.

It will not be necessary for us in the present investigation to consider at all whether the fluid be compressible or incompressible; and we may avoid all restriction by supposing the boundary to be a varying surface S. This will be expressed by considering t, the time, as a variable parameter in the equation of S, which we may therefore assume to be

$$F(x, y, z, t) = 0 \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (a).$$

To express the fact that every particle of the fluid remains on the same side of this surface, or that there is no *flux* across it, we must find the normal motion of the surface, at any part, in an infinitely small time dt, and equate this to the normal component of the motion of a neighbouring fluid particle during the same time. The normal motion of the surface will be the projection upon the normal of the line PP', joining two points P and P' infinitely near one another, of which the former is on the surface S, at the time t, and the latter on the altered surface, at the time $t + dt$, their

marked that cases may actually occur in which this condition is violated; but we cannot infer, as he and subsequent authors have done, that the differential equation is liable to exception in its applications, although we may conclude that the demonstration they have given fails in certain cases. In the demonstration given in the text, founded on a *necessary* condition, there are no such cases of failure. See Poisson, *Traité de Mécanique*, vol. II. No. 652; Duhamel, *Cours de Mécanique, Partie deuxième*, p. 266; Stokes, *Cambridge Philosophical Transactions*, vol. VIII. p. 299.

coordinates therefore satisfying the equations

$$\left.\begin{aligned} F(x, y, z, t) = 0 \\ F(x', y', z', t + dt) = 0 \end{aligned}\right\} \ldots\ldots\ldots (b).$$

Now, with the notation

$$\{F'(x)\}^2 + \{F'(y)\}^2 + \{F'(z)\}^2 = R^2,$$

we have, for the projection of PP' on the normal, the expression

$$\frac{(x' - x)\, F'(x) + (y' - y)\, F'(y) + (z' - z)\, F'(z)}{R}.$$

But, since $x' - x$, $y' - y$, $z' - z$ are infinitely small, we deduce, from equations (b),

$$(x' - x)\, F'(x) + (y' - y)\, F'(y) + (z' - z)\, F'(z) + dt\,.\,F'(t) = 0.$$

Consequently we have, for the normal motion of the surface, during the time dt,

$$-\frac{F'(t)}{R}\, dt.$$

Now the normal component of the motion of a fluid particle in the neighbourhood of the point (x, y, z), during the time dt, is equal to

$$udt\,.\,\frac{F'(x)}{R} + vdt\,.\,\frac{F'(y)}{R} + wdt\,.\,\frac{F'(z)}{R},$$

and we therefore have

$$\frac{udt\,.\,F'(x) + v\,dt\,.\,F'(y) + w\,dt\,.\,F'(z)}{R} = -\frac{F'(t)\, dt}{R}.$$

This may be reduced to the simpler form

$$u\, F'(x) + v\, F'(y) + w\, F'(z) + F'(t) = 0 \ldots\ldots (1),$$

which is the required differential equation of the bounding surface.

To illustrate the applicability of this equation to the class of cases considered by Poisson as exceptional,* let us, for

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* The following passage, extracted from Art. 652 of the *Traité de Mécanique*, contains Poisson's statement with reference to the restriction. "Dans les mouvements des fluides que l'on soumet au calcul, on a coutume de supposer que les points qui se trouvent, à une époque déterminée sur une paroi fixe ou mobile, ou qui appartiennent à la surface libre d'un liquide, demeureront sur cette paroi, ou appartiendront à cette surface pendant toute la durée du mouvement; en sorte que l'on exclut les mouvements très compliqués dans lesquels des points d'un fluide, après avoir appartenu à sa superficie, rentreraient dans l'intérieur de la masse, ou réciproquement; et l'on exclut même les cas ou des points d'un liquide

simplicity, suppose the fluid to be contained within a fixed surface *S*. The condition on which the preceding demonstration depends may, in this case, be satisfied in three different ways by the motion of a fluid particle which at some instant is in contact with the bounding surface.

First, if its line of motion lie entirely on the surface, *S*;

Secondly, if its line of motion lie within *S*, but touch it in one or more points;

Thirdly, if its velocity as it approaches the surface diminish continually, and vanish when it reaches the surface, in which case the line of motion may meet the surface at any angle.

The second and third kinds of motion (which are excluded according to the restrictions already referred to) may be illustrated by taking as examples, actual cases of motion such as the following.

Let part of the fixed bounding surface *S* be plane, and let a certain cylindrical portion of the fluid, touched along a straight line by this plane, revolve continually about its axis, while the rest of the fluid moves within the surface *S* in any way that would be possible were the cylindrical portion a fixed solid; or, whatever be the form of *S*, let a spherical portion of the fluid revolve in any way, touching *S* in a point, the rest of the fluid moving as if this portion were solid. Thus we have two easily imagined cases of motion, in one of which, at a certain straight line, and in the other at a certain point, particles of the fluid are continually *coming to the surface, and then retreating into the interior of the fluid.*

To illustrate the third kind of motion, a certain portion of the fluid may be supposed to move as if it were solid, with a continually decreasing velocity, till it comes to rest, when one point of it reaches the surface; or, we may suppose the portion of the fluid which moves as if it were solid, to be of such a form that a finite portion of its surface may be made to coincide with part of *S*, and we may suppose it to move with continually decreasing velocity, till, when this coincidence takes place, it comes to rest. In such cases particles of the fluid which were originally in the interior, come to occupy positions on the bounding surface; yet the

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passeraient alternativement de la surface libre à la surface en contact avec une paroi fixe ou mobile. Ces conditions particulières auxquelles ou assujettit les mouvemens que l'on considère, s'expriment par les équations suivantes." This passage is followed by a demonstration of the differential equation of the surface.

condition, on which the demonstration given above depends, is not violated; and the equation arrived at does not become nugatory in form, so that we must consider it as literally satisfied.

With reference to the additional restriction made by Poisson, excluding cases in which particles of a liquid *pass alternately from the free surface to the surface in contact with a fixed or moveable vessel containing it,* it may be remarked that, although there will in general be the same kind of difficulty in such cases of motion, as in the others considered above, yet even Poisson's investigation is perfectly applicable, provided that the free and constrained surfaces have a common tangent plane along their line of intersection, a condition which may be actually ensured by arbitrarily constraining, by means of a flexible and extensible envelope, the motion of a band of the bounding surface of the liquid, contiguous to the free surface.

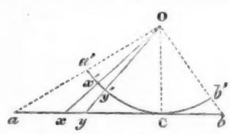
Glasgow College, Jan. 10, 1848.

MATHEMATICAL NOTES.

I.—*On the Attraction of a Straight Line.**

By FERDINAND JOACHIMSTHAL, Berlin.

LET ab be a straight line attracting the point o by Newton's law; draw the perpendicular line oc and describe about o as centre the arc of the circle $a'b'$, touching in c, and comprised between the lines oa and ob; then the attraction of the arc $a'cb'$ has the same magnitude and direction as that of the line ab.



Let us take two points x, y, of the straight line infinitely near; then, denoting the angle xoy by ω, we have two expressions for the double area of the triangle xoy, viz.

$$\overline{oc} \,.\, \overline{xy} = \overline{ox}^2 \,.\, \omega ;$$

therefore
$$\frac{\overline{xy}}{\overline{ox}^2} = \frac{\omega}{\overline{oc}} ;$$

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* This article was communicated to me by the author, when neither of us knew the elegant theorem it contains, with reference to the attraction of a straight line, to have been previously published. I have since found that it is given in the Treatise on Attractions in Earnshaw's *Dynamics;* but the application regarding the position of equilibrium of a point attracted by the perimeter of a triangle is, I believe, new.

but $$\frac{\omega}{oc} = \frac{oc\,.\,\omega}{\overline{oc}^2} = \frac{x'y'}{\overline{ox'}^2};$$

hence $$\frac{xy}{\overline{ox}^2} = \frac{x'y'}{\overline{oc}^2}.$$

The theorem is now proved for two corresponding elements of the arc and the line, and therefore for two corresponding finite parts. We may conclude the following theorem:

The attraction of the perimeter of a triangle vanishes if the attracted point is the centre of the inscribed circle.

Geneva, Aug. 18, 1847.

[Another very interesting application may be made to determine the "lines of force" and "surfaces of equilibrium," due to the attraction of a straight line. Thus, since the arc $a'cb'$ is symmetrical on the two sides of the line bisecting it, it follows that the resultant force on o bisects the angle aob, and therefore touches the hyperbola, and cuts at right angles the ellipse, described through o, from a and b as foci. Hence the lines of force are the series of hyperbolas which have a and b as foci, and the surfaces of equilibrium are generated by the revolution of the confocal ellipses about ab as axis—a known result (proved in Green's *Essay on Electricity*).

We may farther prove, by a simple application of the preceding theorem, that the lines of force due to the attraction of two infinitely long rods in the line ab produced, one of which is attractive and the other repulsive, are the series of ellipses described from the extremities, a and b, as foci, while the surfaces of equilibrium are generated by the revolution of the confocal hyperbolas.]

Glasgow College, Jan. 17, 1848.

II.—*Elementary Investigations of the Methods of Drawing Tangents to the Conic Sections.*

By A. R. Grant, B.A., Fellow of Trinity College, Cambridge.

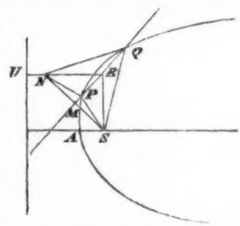
In the ordinary applications of the method of limits to the demonstration of the properties of tangents to the conic sections, the proofs are made to depend on the ultimate equality of triangles, which vanish entirely in the limit. This method presents great difficulties to the beginner, and indeed often leads to an idea that a mere approximation, and not a rigorous proof, is presented to him. It may be added that Newton, in his 7th Lemma, and elsewhere, has taken peculiar pains to exhibit the limiting case in a finite form.

With this object in view, I have been led to the following demonstrations, and I hope that, if they do not answer the end proposed, they will at any rate suggest some method by which it may be attained.

1. *Parabola.* The tangent at any point bisects the angle

between the focal distance, and a perpendicular from the point of contact upon the directrix.

Let a straight line MPQ cut the parabola in two contiguous points P, Q.



Draw $SM \perp MPQ$, and produce it to N; making $MN = MS$.

Through N draw NR parallel to the axis, meeting MPQ in R, and the directrix in U. Join SR, SP, SQ, PN, QN.

The triangles SRM, MRN, are evidently similar and equal. Therefore SR, RN, make equal angles with MPQ, and $SR = RN$. We have evidently $SP = PN$, $SQ = QN$, therefore N is the point of intersection of two circles whose centres are P and Q, and which touch the directrix; therefore N must lie between the directrix and the curve. Therefore

$$RU > RN, \text{ and therefore } > RS;$$

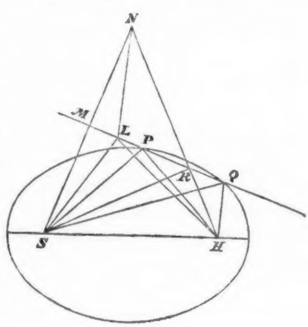
therefore R lies in that part of the secant which is within the curve, and therefore between P and Q. (Wallace's *Conics*, *Parabola*, Prop. I.).

Therefore, in the limit, when Q coincides with P, the point at which a straight line from the focus, and one drawn perpendicular to the directrix, make equal angles with the secant, must coincide with each of them.

Therefore the proposition above enunciated is true.

2. *Ellipse.* The tangent at any point makes equal angles with the focal distances.

Let MPQ be a straight line cutting the curve in two points P, Q, on the same side of the major axis.



Join SP, PH; SQ, QH. Draw $SM \perp MPQ$, and produce it to N, making $MN = MS$. Join NH, cutting MPQ in R, and join SR.

The triangles MRS, MRN are evidently similar and equal, therefore $\angle HRQ = \angle MRN = \angle MRS$; i.e. SR, HR, make equal angles with MPQ. Also if to L, any other point in MPQ, we draw SL, HL,

and join NL, it is evident that the sum of SR, RH, which is equal to NH, is less than the sum of SL, LH, which is equal to the sum of NL, LH.

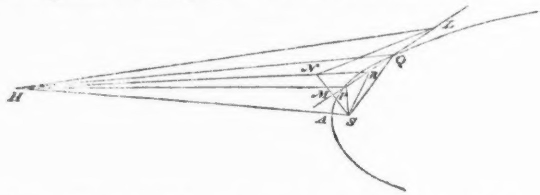
Hence the sum of SR, RH, is less than the sum of any other two lines, drawn from S and H to a point in MPQ.

Therefore the sum of SR, RH, is less than the sum of SQ, QH, and SP, PH, which are each equal to the major axis, and therefore R is a point in that part of the secant which falls within the curve (Wallace, *Ellipse*, Prop. I.), and therefore between P and Q.

Therefore, in the limit, when Q coincides with P, the point at which a straight line drawn to the foci makes equal angles with the secant, coincides with each of them. Therefore the proposition is true.

3. *Hyperbola.* The tangent at any point bisects the angle between the focal distances.

Let MPQ be a straight line cutting the curve in two contiguous points P, Q; join SP, PH; SQ, QH.



Draw $SM \perp MPQ$, and produce it to N; making $MN = SM$. Join HN, and produce it to meet MPQ in R. Join SR. The triangles SRM, MRN, are evidently similar and equal. Therefore $\angle HRM = \angle SRM$, or MPQ bisects the angle SRH. Draw SL, HL, to any point L of MPQ. HN is evidently the difference of HR, SR;

$$\begin{aligned} \text{and} \quad & HN + NL > HL, \\ \text{therefore} \quad & HN > HL - NL \\ & > HL - SL, \quad \text{since } SL = NL; \end{aligned}$$

therefore the difference between HR, SR, is greater than that between any other two lines drawn from the foci to a point in MPQ. But the difference of HQ, SQ = difference of HP, SP; therefore each is less than that of HR, SR, and the point R lies in that part of the secant which is within the curve (Wallace, *Hyperbola*, Prop. I.), and therefore between P and Q.

Hence, in the limit, when Q coincides with P, the point at which straight lines to the foci make equal angles with the secant, coincide with each of them.

Therefore the tangent at P bisects the angle SPH.

ON A PRINCIPLE IN THE THEORY OF SURFACES OF THE SECOND ORDER, AND ITS APPLICATION TO M. JACOBI'S METHOD OF GENERATING THE ELLIPSOID.

By R. Townsend.

[*Continued from* p. 28.]

If the foot of the perpendicular let fall from any point assumed arbitrarily in space upon the polar plane of that point with respect to any surface of the second order, or from any point assumed arbitrarily in the plane of a conic upon any plane whatever passing through the polar line of that point with respect to the conic, be made the vertex of a cone, either enveloping the surface in the former case, or subtending the curve in the latter case, then will the perpendicular itself be always one of the axes of that cone. This is obvious, for, in either case, through the perpendicular let any plane be drawn intersecting the cone and the perpendicular plane, we shall then have in that intersecting plane four right lines, viz. the perpendicular, the intersection with the perpendicular plane, and the two sides of the cone, which will obviously form always a harmonic pencil; but of these the two first are always at right angles to each other, therefore the two latter, their harmonic conjugates, always (from the nature of an harmonic pencil) make equal angles with the perpendicular, which is therefore always an axis of the enveloping or subtending cone.

If in the above the curve be a focal conic of the surface, then, since from the preceding theorem (page 28) the perpendicular let fall upon any plane from its pole with respect to the surface, coincides always with the perpendicular let fall upon the same plane from the focal pole of its intersection with the plane of any focal conic, we see from the property just stated that, as in curves of the second order, the two tangents drawn from any point assumed arbitrarily in the plane of a conic, make always equal angles with the two lines drawn from the same point to either pair of foci, real or imaginary, of that curve; so, in surfaces of the second order, the cone which, from any vertex assumed arbitrarily in space, envelopes any surface of the second order, is always coaxal with the cone which from the same vertex subtends a focal conic, real or imaginary, of that surface. And from the once established fact of these two cones being always coaxal, there is no difficulty whatever in shewing, from the simplest geometrical considerations, that they are moreover always confocal. (See *Journal*, New Series, Vol. II. p. 40.)

Since (as we have seen) every two surfaces whatever of the second order which intersect in two plane curves, real or imaginary, have always the same polar plane for every point on the line of intersection of the two planes, if that line intersect a dirigent cylinder, that is, if it intersect a directrix of either surface, then will the polar plane of the point of intersection with respect to that surface, and therefore with respect to the other, pass always through the focus corresponding to that directrix, and be always perpendicular to the right line joining the point and focus: that joining line is therefore the perpendicular let fall from a point upon its polar plane with respect to either of the surfaces, and the focus is the foot of that perpendicular. Hence we see that

If through any point on any directrix of a surface of the second order, any two planes whatever be drawn intersecting the surface, then will the cone which from the corresponding focus envelopes any surface whatever of the second order, passing through the two curves of section, have always for one of its axes the line connecting that focus with the point on the directrix.

This very general theorem obviously contains under it the following interesting particular cases, it being remembered that when any two planes come together and coincide in one, then will each obviously pass always through every point and through every line contained in the other.

If any two surfaces whatever of the second order envelope each other, then will the cone which from any focus of either surface envelopes the other, have always for one of its axes the line joining that focus with the point where the plane of contact intersects the corresponding directrix.

Hence, in any given surface of the second order, to find by a geometric construction an axis of the cone which from any focus envelopes any inscribed or circumscribed surface of the second order, we have obviously but to connect the given focus with the point where the plane of contact of the enveloping surface intersects the corresponding directrix, and the connecting line will then be the axis required.

In the particular case when the enveloping surface flattens down into a plane, then the enveloping cone becomes obviously the cone subtending the plane section of the given surface. Hence, in any surface of the second order, to find by a geometric construction an axis of the cone which from any focus subtends any plane section of the surface, we have but to connect that focus with the point where the plane

of the section intersects the corresponding directrix, and the connecting line will then be the axis sought.*

In the particular case when the given surface of the second order is an *umbilicar* surface of revolution, then obviously is *every* line in either of the two dirigent planes corresponding to the two foci indifferently a directrix of the surface corresponding to the same focus. Hence, since every right line in a plane intersects every other in the same plane, and since at every point it intersects some other in the plane, we see from the above that

If through any right line taken arbitrarily in either of the two dirigent planes of an umbilicar surface of revolution of the second order, any two planes whatever be drawn intersecting the surface, then will the cone which from the corresponding focus envelopes any surface whatever of the second order passing through the two curves of section, be always such that *every* right line drawn from the focus to the assumed line in the dirigent plane, will be indifferently an axis of the cone. Hence we see that that cone will be always of revolution, and that its equatoreal principal plane will be always the plane containing the focus and the assumed line in the dirigent plane.

If the two planes come together and coincide in one, then, whatever be the position of that one, the line in which it intersects either dirigent plane must always intersect at every point a directrix of the surface. Hence we see that

If any surface whatever of the second order be inscribed in, or circumscribed to, any umbilicar surface of revolution of the second order, then will the cone which from either focus of the latter surface envelopes the other, be always a cone of revolution, of which the plane containing the focus and the line in which the plane of contact of the enveloping surface intersects the corresponding dirigent plane, will be always the equatoreal principal plane.†

In the particular case when the enveloping surface flattens down into a plane, we get the old and often established theorem, that the cone which from either focus of an umbilicar surface of revolution of the second order subtends any plane section of the surface will be always a cone of revolution, of which the plane containing that focus and the line in which the plane of the section intersects the

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* This last theorem is due to Professor MacCullagh.

† This theorem is due to Professor Chasles.

corresponding dirigent plane will be always the equatoreal principal plane.

And finally, when the plane of the section passes through the focus, and when therefore the cone also flattens down into a plane, then, since in that limiting state it remains still of revolution, we get the obvious property, that every section of an umbilicar surface of revolution of the second order, whose plane passes through either of the foci of the surface, will have that point always for one of its own foci, and the line in which its plane intersects the corresponding dirigent plane for the directrix corresponding to that focus.

When any two surfaces whatever of the second order have double contact, real or imaginary, then obviously will every plane passing through the real chord of double contact intersect them in a pair of conics which also will always have double contact, real or imaginary, the two points and the chord of contact being always the same as in the surfaces themselves.

Hence, since every section of a sphere is always a circle, and since every section of an evanescent sphere is always an evanescent circle, we see that

Every plane passing through any focus of a surface of the second order, modular or umbilicar, and through the corresponding directrix, will always intersect the surface in a conic of which that point and line will be always a focus and corresponding directrix: or, to state the same property under another and better known form, every plane normal at any point to a focal conic of any surface of the second order will intersect the surface in a conic of which the focal point will be always a focus, and of which the directrix of the surface corresponding to that focus will be always the corresponding directrix.

Of this theorem the plane sections of a cone or cylinder perpendicular to their focal lines, the meridians of a *modular* surface of revolution, and *all* plane sections passing through a focus of an *umbilicar* surface of revolution, afford well-known and familiar instances.

The last case is also evident from the obvious consideration, that if any two surfaces whatever of the second order, in place of having merely double contact, actually envelope each other, then will *every* plane drawn arbitrarily in any direction intersect them in a pair of conics which will always have double contact, real or imaginary, the chord of which will always be the real right line in which the plane of section intersects the plane of contact; and that therefore, if one of the surfaces be a sphere and the other consequently of revolu-

tion, then will the two curves of section be always a conic and circle having double contact, real or imaginary: but a focus of an umbilicar surface of revolution is obviously the evanescent limit to a sphere, not only having double contact with, but actually inscribed in, the surface, the plane of imaginary double contact being the dirigent plane of the surface corresponding to that focus. Hence, in that class of surfaces of revolution each focus is always the evanescent limit to a circle having double imaginary contact with *every* section whose plane passes through it, and the chord of double contact is always the line in which that plane intersects the corresponding dirigent plane, and therefore that point and line are always respectively a focus of the section and its corresponding directrix.

It is a known property of surfaces of the second order, that if any two surfaces be both circumscribed to or inscribed in a third, their curve of intersection will always break up into two plane curves, both of course of the second order, and either or both real or imaginary, as the case may be; the two planes of these curves passing always both through the intersecting line of the two planes of contact, and the four planes forming always an harmonic system.*

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* Of this theorem Mr. Salmon has given the following elegant and ingenious analytical solution: If $S = 0$ be the equation of any surface of the second order, and $A = 0$, $B = 0$ those of any two planes, then will the whole system of surfaces of the second order which pass through the two conics of section be obviously included in the equation $S + \alpha AB = 0$, α being a variable parameter.

Hence, if $A=0$, $B=0$, be the planes of contact of any two surfaces of the second order enveloping $S=0$, then will the equations of these two surfaces be $S + \mu^2A^2 = 0$ and $S + \nu^2B^2 = 0$, μ and ν being arbitrary, but constant. Subtracting these, we get, as one of the surfaces of the second order passing through the intersecting curve of the two enveloping surfaces, $\mu^2A^2 - \nu^2B^2 = 0$, a surface obviously resolvable into $\mu A + \nu B = 0$, and $\mu A - \nu B = 0$, that is, into two planes passing both through the line of intersection of A and B, lying (on account of the opposite signs of the parameters) in different dihedral angles, and forming with A and B a faisceau of four planes of which the anharmonic ratio is unity; for, $-\frac{\mu}{\nu}$ being the ratio of the perpendiculars let fall from any point of the plane $\mu A + \nu B = 0$, upon the planes $A = 0$, and $B = 0$, is the ratio with its proper sign of the trigonometrical sines of the angles it makes with these planes; and for the same reason, $+\frac{\mu}{\nu}$ is the ratio with its proper sign of the sines of the angles with the same planes by the other plane $\mu A - \nu B = 0$.

The other theorem in the text immediately succeeding is obviously the polar reciprocal of this, and of both a very simple and elementary geometrical solution had been previously given by M. Poncelet.

With respect to the equation $S + \alpha A^2 = 0$, which in general represents any surface of the second order enveloping $S = 0$, along its curve of section

And again, that their common circumscribing developable will, under the same circumstances, always break up into two cones, both of course also of the second order, and either or both real or imaginary, as the case may be; the vertices of the two cones lying always both on the line joining the poles of the two planes of contact with respect to the circumscribing or circumscribed surface, and the four points forming always a harmonic system. Hence, conversely, if any two surfaces of the second order intersect in a pair of plane curves, real or imaginary, they will always admit of an infinite number of common enveloping surfaces of the second order, and amongst them of two enveloping cones, either or both real or imaginary, as the case may be.

If now one of the two surfaces be a sphere, having double contact with, and therefore intersecting the other in two plane curves, real or imaginary, the two common enveloping cones, as being then both circumscribed to a sphere, must obviously be always both of revolution. And if, finally, the sphere dwindle into a point, then moreover will these two cones of revolution obviously come together and coincide in one with its vertex at that point. Hence we see that the cone, real or imaginary, which from any point on either of its focal conics envelopes a surface of the second order, will be always of revolution.

Again, in the general state of the sphere of double contact, the two planes of contact of the two common enveloping cones of revolution with respect to the sphere, and also their two planes of contact with respect to the surface, always pass obviously all four through the common chord of double contact; therefore in the limiting and evanescent state of that sphere, when the two cones come together and coincide in one with its vertex at a focus, the plane passing through that focus and its corresponding directrix will be the polar plane with respect to the infinitely small sphere of the vertex of that cone—but the polar plane of any point with respect to any sphere is always perpendicular

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with $A = 0$, Mr. Salmon has also very ingeniously remarked, that if the particular value assigned to α be such as to make the enveloping surface pass through the pole of the plane A with respect to the surface S, then must that surface be necessarily a cone; and that therefore if $A = 0$ be the equation of the polar plane of any given point $x'y'z'$ with respect to any given surface of the second order, represented by $S = 0$, then always will $SA'^2 - S'A^2 = 0$ be the equation of the cone which from that point as vertex envelopes that surface,—an equation which may therefore in all cases be immediately formed when we have the equation of the surface and the coordinates of the vertex.

to the axis of revolution of the cone which from that point envelopes the sphere. Hence we see that the axis of revolution of the cone, real or imaginary, which from any focus as vertex envelopes a surface of the second order, is always perpendicular to the plane which contains that focus and its corresponding directrix, and therefore always coincides with the focal tangent at that point.

Returning again to the sphere in its general state of double contact with the surface, if we conceive any third surface of the second order to be described having double contact with the two at the same two points, real or imaginary, then, for the same reason as before, will the sphere and the new surface also admit always of a pair of common circumscribing cones of revolution: so that, hence, if as before we conceive the sphere to dwindle into a point, and the two cones therefore to come together and coincide in one with its vertex at that point, we see that if any surface of the second order intersect another in a pair of plane curves, such that the two planes intersect in any directrix of either surface, then will that line be also a directrix of the other surface; the focus in the former corresponding to that directrix, will be also the corresponding focus in the latter; the plane passing through that focus perpendicular to the common directrix will be a principal plane common to the two surfaces; their two focal conics in that principal plane, besides passing obviously both through the common focus, will moreover have the same focal tangent at that point, the perpendicular to the plane containing focus and directrix; and the two cones which from that common focus envelope the two surfaces will be always both of revolution, and will moreover have always a common axis of revolution, the common focal tangent at their common vertex.

This last property is very general, and leads to several other very general results: of these, the few following are evident. If through any directrix whatever of a surface of the second order, any two planes be drawn arbitrarily intersecting the surface in a pair of conics, real or imaginary, then will *every* surface of the second order passing through the two plane curves of section have always that directrix, the corresponding focus, and the corresponding principal plane common with the original surface; its focal conic in that plane will always pass through the common focus, and there touch the focal conic of the original surface in the same plane, and its enveloping cone from that focus as vertex will be always of revolution, and will always have the same axis of revolution,

viz. the common focal tangent, or the perpendicular at the common focus, to the plane containing that point and the corresponding directrix.

When two planes come together and coincide in one, then each obviously passes always through every line contained in the other: hence, from the same, we see that if through any directrix whatever of a surface of the second order any plane be drawn arbitrarily intersecting the surface, then will every surface of the second order, inscribed in or circumscribed to the surface along the curve of section, have that directrix, its corresponding focus, and the corresponding principal plane common with the original surface; one of its focal conics will always pass through that corresponding focus, and there touch the original focal conic, and its enveloping cone from that focus as vertex will be always of revolution, and will have always a common axis of revolution, viz. the perpendicular to the plane containing the common focus and corresponding directrix, that is the common focal tangent.

In the particular case when the inscribed or circumscribed surface becomes infinitely flat, that is, when it becomes either portion of the intersecting plane bounded by the conic of section, and when therefore the enveloping cone becomes the cone subtending that section, we see that the cone which from any focus of a surface of the second order subtends any section of the surface, whose plane passes through the corresponding directrix, will be always of revolution, and will always have the same axis of revolution, viz. the focal tangent at that focus, or the perpendicular to the plane containing that point and the corresponding directrix.*

In the last case, if the plane passing through the directrix pass also through the focus, the cone will obviously degenerate into the plane of the section—but in this limiting state, being still always of revolution, we see that its vertex must be the focus of the section: this is the property which was proved on other principles a little further back.

If the original surface of the second order be an umbilicar surface of revolution, then will *every* line in either of its two dirigent planes be indifferently a directrix of the surface to the same corresponding focus: hence, from the present general theorem we also deduce at once as particular cases the same focal properties of that class of surfaces of revo-

* This last theorem is due to Professor MacCullagh.

lution which a short while since we previously deduced as particular cases of another and wholly different general theorem, viz., If through any line taken arbitrarily in either dirigent plane of an umbilicar surface of revolution of the second order, any two planes whatever be drawn arbitrarily intersecting the surface in a pair of conics, real or imaginary, then will the cone which from the corresponding focus envelopes any surface whatever of the second order passing through the two curves of intersection be always of revolution, and its axis of revolution will be always the perpendicular through that focus to the plane containing that point and the assumed line in the dirigent plane: and again, the cone which from either focus envelopes any surface whatever of the second order, arbitrarily inscribed in or circumscribed to the original surface, will be always of revolution, and its axis of revolution will be always the line through that focus perpendicular to the plane containing that point, and the line in which the plane of contact of the enveloping surface intersects the corresponding dirigent plane: and finally, the cone which from either focus subtends any plane section of the surface will be always of revolution, and its axis of revolution will be always the perpendicular through that focus to the plane containing that point, and the line in which the plane of the section intersects the corresponding dirigent plane.*

* If any surface of revolution of the second order, modular or umbilicar, be circumscribed to or inscribed in any other surface whatever of the second order, its axis of revolution must always lie in a principal plane of the latter: for connecting the vertex of the cone which envelopes both surfaces along the curve of contact with the centre of that curve, the joining line will of course pass through the centres of the two surfaces, and the plane passing through that line perpendicular to the plane of contact will be a principal plane of the surface of revolution, and will therefore bisect at right angles all the chords of the curve perpendicular to its intersection with the plane of contact: but that system of chords belongs also to the other surface, and the perpendicular plane passes through the centre of that surface; hence, whether that plane be the equatoreal or a meridian plane of the surface of revolution, it coincides with a principal plane of the other surface, and therefore in either case the axis of revolution lies in a principal plane of the latter.

Hence, and from the two last of the theorems above, we see at once that to every surface whatever of the second order there can be always inscribed or circumscribed three different and distinct systems of surfaces of revolution of the second order, of which two will be always umbilicar and the third always modular, and of which the two plane curves loci of the two systems of real foci of the former will be always the two real focal conics of the enveloped surface, while the single plane curve locus of the unique system of imaginary foci of the latter will be always the single imaginary focal conic of the same surface. And again, that through every given conic there can be always sent two different and distinct systems of surfaces of revolution of the second order, of which one will

The second of these focal properties of umbilicar surfaces of revolution, which was originally given by Professor Chasles as an extension of the third, which was well known and had been often established, is immediately reducible directly to the latter: for the cone which from any vertex whatever envelopes either of any two enveloping surfaces of the second order will always intersect the other in a pair of plane curves, inasmuch as that cone and the latter surface both at the same time envelope a third surface (*Note*, page 101), and therefore the cone which from an umbilicar focus of a surface of revolution envelopes any inscribed surface will always subtend a plane section of the original surface.

Taking any two surfaces whatever of the second order which envelope each other, let any transversal plane be drawn parallel to a cyclic plane of either, but with that single restriction otherwise arbitrary in absolute position, that plane will then obviously always intersect the surfaces in a conic and a circle respectively, which will always have double contact, real or imaginary, the chord of double contact being always the line of intersection of the transversal plane with the plane of contact of the enveloping surfaces.

Conceive now the cutting plane to move parallel to itself, until that the circular section of the surface to whose directive plane it is parallel, diminishing in magnitude becomes just evanescent, then will that plane be a tangent plane to that surface at one of its umbilici, and the above property holding of course in that limiting as well as in the general case, we hence, from the same general principle, see at once that the tangent plane at an umbilicus of either of any two enveloping surfaces of the second order will intersect the other in a conic of which that point will be always a focus, and of which the intersection of the plane with the plane of contact of the surfaces will be always the corresponding directrix.

If one of the enveloping surfaces be a sphere, the other will of course be of revolution, and every point on the former will indifferently be an umbilicus: hence we see that if a sphere be inscribed in any surface of revolution of the second order, then will the point of contact of every tangent plane to the sphere be always a focus of the section that

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be always umbilicar and the other always modular, and of which the curve locus of the real system of foci of the former will be always the real conic focal to the given, while that of the imaginary system of foci of the latter will be always its imaginary focal.

plane determines in the surface, and the intersection of the same plane with the plane of contact will be always the corresponding directrix.

Under this last comes obviously a very ancient theorem, viz., Every sphere placed on a horizontal plane and exposed to the light of the sun, stands always in the focus of the contour of its own shadow whether of umbra or penumbra, the plane of the circle of light and partial darkness, or of partial and total darkness, always intersecting the horizontal plane in the corresponding directrix.

The above general property of the umbilici of any two enveloping surfaces of the second order may also be directly and very simply established, without the aid of the principle of the evanescent circle of double contact, though it affords a good example illustrative of the utility of that principle. For if in any surface of the second order in which another is inscribed we conceive any third surface to be also inscribed, this last will always intersect the original inscribed surface in a pair of plane curves, of which if one be a circle for either surface, it will of course be also a circle for the other, and if it be an evanescent circle for either, it will be an evanescent circle for the other: hence we see that if the new surface touch the original inscribed surface at an ombilic of the latter, its point of contact must at the same time be also one of its own umbilici, and therefore in the particular case when it flattens down into a plane, that point of contact must always be one of its foci.

Many other examples might easily be brought forward illustrative of the utility and power of the same general principle, as a method for investigating the focal properties in surfaces of the second order; but as the extent of that subject precludes the possibility of following it out in detail, we need give no more for the mere purpose of illustrating that particular employment of the principle. But, as we said before, there is another and not less extensive sphere of application, in which the same principle may be no less usefully employed in a manner altogether different, viz. as a test for ascertaining whether, in unknown or undetermined cases, certain points which enjoy known properties with respect to a surface of the second order, or which occupy an important position in the generation of such a surface, be or be not focal points of that surface, and if they be, whether they be modular or umbilicar foci. And in this as in the former employment of the principle, many examples might also without much difficulty be brought forward,

illustrative of its equal power and utility: of these we select one in consequence of the historical interest connected with the circumstances attending it, and thus proceed to one of the expressed objects, though not perhaps the most useful part, of the present paper: in the first place, to establish a theorem connected with M. Jacobi's method of generating surfaces of the second order, which, so far as the author is aware, has not hitherto been proved, and which moreover, so far as he can see, though it appears immediately from the present principle, would be rather difficult to establish by any of the ordinary methods: and in the next place, to shew that the method of generation proposed by that illustrious mathematician, for the purpose of establishing a particular and interesting analogy between curves and surfaces of the second order, is implicitly contained in Professor MacCullagh's, or the modular method of generating the same surfaces.

[*To be continued.*]

Trinity College, Dublin, February 1847.

ON THE VALUES OF A PERIODIC SERIES AT CERTAIN LIMITS.

By Francis W. Newman.

The discussions which from time to time arise concerning Fourier's theorem and its simplest cases, shew that it is not yet superfluous to exhibit elementary and rigorous proof of these. The following process is that of Fourier himself, except that he has left it to his reader to apply it at the limits themselves. It seems instructively to shew how erroneous it is to assert, that in the algebra, "what is true *within* the limits is true *at* the limits."

Let two cardinal series be considered,

$$\cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \tfrac{1}{7}\cos 7x + \&c.\ldots\ldots$$

$$\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x + \&c.\ldots\ldots$$

which shall first be treated as containing a finite number of terms. Put Y, Z for the sums of $2n$ terms; and after it shall have been ascertained that the series converge when $n = \infty$, let y, z represent the sums *ad infinitum*.

$$\text{Then } \frac{dY}{dx} = -\{\sin x - \sin 3x + \sin 5x - \ldots - \sin(4n-1)x\}$$

$$= \frac{\sin 4nx}{2\cos x}, \text{ by ordinary summation,}$$

and $\dfrac{dZ}{dx} = \cos x - \cos 2x + \cos 3x - \ldots - \cos 2nx,$

$$= \frac{\cos \frac{1}{2} x - \cos \frac{1}{2}(4n+1)x}{2 \cos \frac{1}{2} x}.$$

Integrating both by parts, we get

$$Y = C_n - \frac{\cos 4nx}{4n \cdot 2 \cos x} + \frac{1}{8n} \int \cos 4nx \,.\, d \sec x,$$

$$Z = c_n + \tfrac{1}{2}x - \frac{\sin \frac{1}{2}(4n+1)x}{(4n+1)\cos \frac{1}{2}x} + \frac{1}{4n+1} \int \sin \frac{4n+1}{2} x \, d \sec \tfrac{1}{2}x.$$

Whenever the limits are such that the closing integrals are finite, the two terms vanish upon supposing $n = \infty$. But when $\pm x$ reaches $\frac{1}{2}\pi$ in the former or π in the latter, we cannot be sure of this: hence the investigation separates itself into cases.

First, in reference to Y, let ϵ be some positive finite number, however small, and x lie between $\pm(\frac{1}{2}\pi - \epsilon)$; then $\sec x$ is never greater than $\operatorname{cosec} \epsilon$, and $\int \cos 4nx \, d \sec x$ is finite between the finite limits. On the other hand, if we fix 0 as the lower limit of x in $\int$, since 0 is between $\pm(\frac{1}{2}\pi - \epsilon)$, we get

$$Y \text{ or } 1 - 3^{-1} + 5^{-1} - \ldots - (4n-1)^{-1} = C_n - (8n)^{-1} \cos 4nx;$$

which gives a finite value to C_n when $n = \infty$.

This finite value is

$$C = 1 - 3^{-1} + 5^{-1} - 7^{-1} + \&c. \ldots ad\ infin. = \tfrac{1}{4}\pi.$$

Make therefore $n = \infty$, and confine x between $\pm(\frac{1}{2}\pi - \epsilon)$, and we get

$$y = \tfrac{1}{4}\pi \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But this fails of giving us y when $x = \pm \frac{1}{2}\pi$, at which limit Fourier asserts that y has any or every value between $\frac{1}{4}\pi$ and $-\frac{1}{4}\pi$. A proof may be offered as follows.

Put $x = \frac{1}{2}\pi - \dfrac{u}{4n}$, and let u vary from a to 0.

Since $\dfrac{dY}{dx} = \dfrac{\sin 4nx}{2 \cos x}$, $\therefore Y = C_n' + \displaystyle\int \frac{\sin 4nx}{2 \cos x} dx.$

Now $\cos x = \sin \dfrac{u}{4n}$; $dx = -\dfrac{du}{4n}$; $\sin 4nx = \sin(2n\pi - u) = -\sin u$;

therefore $$Y = C_n' + \tfrac{1}{2}\int \sin u \left(\sin \frac{u}{4n}\right)^{-1} \cdot \frac{du}{4n}.$$

To fix C_n', take $u = 0$ as the lower limit of the integral, $x = \frac{1}{2}\pi$, $Y = (2n - 1)$ terms of a series of which every one vanishes; or $Y = 0$. Then $C_n' = 0$.

Next, make $n = \infty$;

therefore $$y = \tfrac{1}{2}\int_0 \frac{\sin u \,.\, du}{u} \quad \ldots\ldots\ldots\ldots\ldots \quad (2).$$

The range of u is up to $u = a$; but a is as arbitrary as u. At the extreme values $a = \pm\infty$, we get $y = \pm\frac{1}{4}\pi$; and by assigning intermediate values to u, we may give to y any proposed value between $\pm\frac{1}{4}\pi$. This justifies Fourier's assertion, that the locus of the curve

$$y = \cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \&c.. .\textit{ad infin.}$$

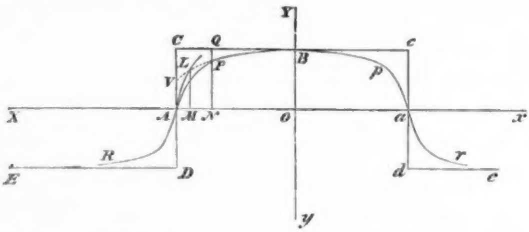
is a connected series of straight lines alternately parallel and perpendicular to the axes.

The explanation of this is given (from Fourier) in that well-known work, De Morgan's *Differential Calculus*.

Imagine the locus of

$$Y = \cos x - \tfrac{1}{3}\cos 3x + \ldots - (4n - 1)^{-1}\cos(4n - 1)\,x,$$

to be constructed, which cannot much differ from the line



RAPBpar, if *XOx*, *YOy* are the axes. Make $n = \infty$, and this locus approximates towards the system of lines *EDCcde* ....which is the true locus of the infinite equation that connects y and x.

This also shews *why* the whole perpendicular line *CD* is included in the locus. Let $ON = x$, $NP = Y$, $NQ = y$. If then we suppose x to be constant, and n to increase perpetually, P runs up towards Q; and if n is constant, but AN diminishes perpetually, P runs towards the single point A. But besides these hypotheses, we are at liberty *to suppose*

AN while vanishing, to be a function of n which is increasing; just as above, we have put $x = \frac{1}{2}\pi - \frac{1}{4}n^{-1}u$. Then P changes with n on to another curve, while N changes, with x, to another point M; or by the double change, P moves as to L, within the area $CAPQ$; and with the progressive increase of n and of x, it may tend towards *any* point V in the line AC. What point shall be V, depends on the function which AN is of n; or, with us, on the highest limit of u.

This may warn us generally, that in estimating the value of a series $y_n = \phi_1 x + \phi_2 x + \ldots + \phi_n x$, taken when $n = \infty$, the result, at singular points, depends on the question, whether x can itself be a function of n. Algebraically and in the abstract this must be looked on as possible: in each physical application the possibility will need to be inquired into.

Next, we may more rapidly dispatch the investigation of Z. Let the integral $\int \sin \frac{1}{2}(4n+1)\, x\, d \sec \frac{1}{2} x$, begin from $x = 0$; then $Z = 0$, since the number of terms is finite, and each separately $= 0$; consequently $c_n = 0$. Farther, let ε be positive, and x lie between $\pm(\pi - \varepsilon)$, which contains the value $x = 0$ by which we defined c_n. Then the unknown integral is finite. Put $n = \infty$, and we have

$$z = \tfrac{1}{2}x, \text{ as long as } x \text{ is between } \pm(\pi - \varepsilon) \ldots \quad (3).$$

But for the limit $x = \pm\pi$, we proceed otherwise.

$$\text{Put } x = \pi - \frac{v}{2n}, \quad Z = c_n' + \tfrac{1}{2}x - \tfrac{1}{2}\int \cos \frac{4n+1}{2} x \sec \tfrac{1}{2}x\, dx,$$

$$\cos \tfrac{1}{2}x = \sin \frac{v}{4n}, \quad \cos \frac{4n+1}{2} x = \sin\left(v + \frac{v}{4n}\right); \quad dx = -\frac{dv}{2n};$$

$$\text{whence} \quad Z = c_n' + \tfrac{1}{2}x + \int \sin\left(v + \frac{v}{4n}\right)\left(\sin \frac{v}{4n}\right)^{-1} \cdot \frac{dv}{4n}.$$

Let the last integral begin from $v = 0$, $x = \pi$, therefore as n is finite, $Z = 0$, and $0 = c_n' + \frac{1}{2}\pi$; afterwards pass to the limit of $n = \infty$; therefore

$$z = \frac{x - \pi}{2} + \int_0 \frac{\sin v dv}{v};$$

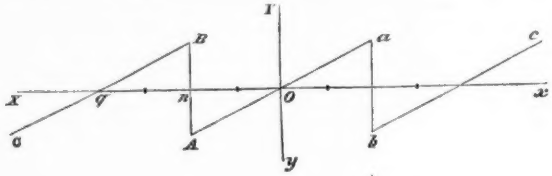
or, since x converges to π, when n increases towards infinity,

$$z = \int_0 \frac{\sin v dv}{v}, \text{ when } x \text{ is at } \pi \ldots\ldots\ldots (4).$$

This, as before, means that z has any value whatever between $\pm\frac{1}{2}\pi$, and shews the locus of

$$y = \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \&c.$$

to be of the zigzag form. . . . *CABabc*. . . .



ON A GENERAL TRANSFORMATION OF ANY QUANTITATIVE FUNCTION.

By George Boole.

There is a view of the general problem of the solution of equations algebraic and transcendental, which I do not remember to have seen taken by any writer on the subject. It is embodied in the following proposition.

If we knew all the transformations of the arbitrary function $f(x)$, we should be able to solve any proposed equation by a general theorem.

This will be obvious if we consider that the solution of an equation is, properly speaking, the transformation of an inverse function into a function in which all the implied operations are direct. Thus the solution of the equation $\phi(u) = x$ is really a transformation of the inverse function $\phi^{-1}(x)$. Lagrange's and Laplace's theorems might be interpreted into results of this nature, although they are generally regarded from a different point of view. I propose here to consider the more general question which is suggested by the above considerations. The most remarkable conclusion to which my inquiries lead is, that the general transformation of $f(x)$ involves a new arbitrary function, particular forms of which conduct us to the theorems of Laplace and Lagrange. The analysis which I shall employ is, in principle, that which I have adopted in a similar investigation, published in the 21st volume of the *Transactions of the Royal Irish Academy*, but it is more simple and more direct.

It somewhat simplifies the analysis to commence with the differential coefficient $f'(x)$ of the function $f(x)$ under consideration.

By an easy transformation of Fourier's theorem, we have

$$f'(x) = 2\pi \iint_{-\infty}^{\infty} da\, dv\, \varepsilon^{(x-a)\, v\sqrt{(-1)}} f(a) \;\ldots\ldots\; (1),$$

provided that x lie within the limits of integration relative to a. At present I suppose those limits undetermined.

Let $a = \phi(\alpha)$, then $da = \phi'(\alpha)\, d\alpha$, and

$$f'(a)\, da = f'\phi(\alpha)\, \phi'(\alpha)\, d\alpha = \frac{d}{d\alpha} f\phi(\alpha)\, d\alpha,$$

and, substituting,

$$f'(x) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} d\alpha\, dv\, \varepsilon^{\{x-\phi(\alpha)\}\, v\sqrt{(-1)}} \frac{d}{d\alpha} f\phi(\alpha) \;\ldots\;(2)$$

$$= \frac{1}{2\pi} \iint_{-\infty}^{\infty} d\alpha\, dv\, \varepsilon^{(x-\alpha)\, v\sqrt{(-1)}}\, \varepsilon^{Av\sqrt{(-1)}} \frac{d}{d\alpha} f\phi(\alpha),$$

provided that $A = \alpha - \phi(\alpha)$.

Let the limits of α be $-\infty$ and ∞, and, expanding the second exponential, we have

$$f'(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\alpha\, dv\, \varepsilon^{(x-\alpha)\, v\sqrt{(-1)}}$$

$$\left\{1 + Av\sqrt{(-1)} + \frac{1}{1.2} A^2 \{v\sqrt{(-1)}\}^2 + \&c.\right\} \frac{d}{d\alpha} f\phi(\alpha) \ldots (3),$$

provided that x lie within the limits of $\phi(\alpha)$. Now for all integer values of n, we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\alpha\, dv\, \varepsilon^{(x-\alpha)\, v\sqrt{(-1)}} A^n \{v\sqrt{(-1)}\}^n \frac{d}{d\alpha} f\phi(\alpha)$$

$$= \left(\frac{d}{dx}\right)^n \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\alpha\, dv\, \varepsilon^{(x-\alpha)\, v\sqrt{(-1)}} A^n \frac{d}{d\alpha} f\phi(\alpha)$$

$$= \left(\frac{d}{dx}\right)^n \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\alpha\, dv\, \varepsilon^{(x-\alpha)\, v\sqrt{(-1)}} \{\alpha - \phi(\alpha)\}^n \frac{d}{d\alpha} f\phi(\alpha)$$

$$= \left(\frac{d}{dx}\right)^n \{x - \phi(x)\}^n \frac{d}{dx} f\phi(x) \ldots\ldots\ldots\ldots\ldots (4),$$

by Fourier's theorem. Reducing by this formula each term of (3), we have

$$f'(x) = \frac{d}{dx} f\phi(x) + \frac{d}{dx}\{x - \phi(x)\} \frac{d}{dx} f\phi(x)$$
$$+ \frac{1}{1.2}\frac{d^2}{dx^2}\{x - \phi(x)\}^2 \frac{d}{dx} f\phi(x) + \&c.;$$

and multiplying both members by dx, and integrating,

$$f(x) = f\phi(x) + \{x - \phi(x)\}\frac{d}{dx} f\phi(x) + \frac{1}{1.2}\frac{d}{dx}\{x - \phi(x)\}^2 \frac{d}{dx} f\phi(x)$$
$$+ \frac{1}{1.2.3}\frac{d^2}{dx^2}\{x - \phi(x)\}^3 \frac{d}{dx} f\phi(x) + \&c.\ldots.(5),$$

which is the theorem sought. The only condition relative to the limits is that the equation

$$\phi(a) = x \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

shall admit of being satisfied by a real value of a for that particular value of x which is employed in the theorem. In other respects the function $\phi(x)$ is arbitrary.

If we should assume $\phi(x) = c$, the theorem (5) would give $f(x) = f(c)$ all the terms after the first vanishing; but the condition (6) gives $x = c$, and shews that it is only for this particular value of x that the theorem (5) is to be supposed true. Thus there is no inconsistency.

If we make $\phi(x) = x - h$, we have $x - \phi(x) = h$, and (5) gives

$$f(x) = f(x-h) + h\frac{d}{dx} f(x-h) + \frac{1}{1.2} h^2 \frac{d^2}{dx^2} f(x-h) + \&c.,$$

which is, by Taylor's theorem, a particular verification.

To apply (5) to the solution of equations, let there be given

$$f(u) = x \ldots\ldots\ldots\ldots\ldots\ldots (7);$$

and writing in (5) f^{-1} for f, since $u = f^{-1}(x)$, we have

$$u = f^{-1}\phi(x) + \{x - \phi(x)\}\frac{d}{dx} f^{-1}\phi(x)$$
$$+ \frac{1}{1.2}\frac{d}{dx}\{x - \phi(x)\}^2 \frac{d}{dx} f^{-1}\phi(x) + \&c.$$

To eliminate the inverse functional sign f^{-1} let us write $\phi(x) = f(x)$, we have then

$$u = x + x - f(x) + \frac{1}{1.2}\frac{d}{dx}\{x - f(x)\}^2 + \&c.\ldots.(8).$$

The condition (6) here becomes $f(a) = x$, and simply indicates that the given equation (7) must have a real root.

Suppose that we have $f(u) = x$, and wish to determine in the most general manner $F(u)$ as a function of x, we have

$$F(u) = Ff^{-1}(x) \ldots\ldots\ldots\ldots (9);$$

write then in (5) Ff^{-1} for f, and for $\phi(x)$, which is arbitrary, write $f\phi(x)$; also, for simplicity, let

$$X = x - f\phi(x) \ldots\ldots\ldots\ldots (10),$$

then we have

$$F(u) = F\phi(x) + XF'\phi(x)\,\phi'(x) + \frac{1}{1.2}\frac{d}{dx}X^2F\phi(x)\,\phi'(x) + \&c. \ldots\ldots\ldots (11).$$

The theorems of Lagrange and Laplace for the expansion of functions are particular cases of the above, which, it is seen, includes an infinite diversity of particular theorems corresponding to the different forms which may be attributed to $\phi(x)$. To deduce Lagrange's theorem, we have

$$u = x + hf(u),$$

therefore $$u - hf(u) = x.$$

Hence in (11), writing $u - hf(u)$ in the place of $f(u)$, or $x - hf(x)$ in that of $f(x)$, we have

$$X = x - \phi(x) + hf\{\phi(x)\}.$$

Let $\phi(x) = x$, then $X = hf(x)$, and therefore

$$F(u) = F(x) + hf(x)\,F'(x) + \frac{h^2}{1.2}\frac{d}{dx}f(x)^2\,F'(x) \ldots (12),$$

which is Lagrange's theorem: but every other form of $\phi(x)$ which makes $\phi(x) - hf\{\phi(x)\}$ real, and therefore at least every real form of $\phi(x)$, will give a true expansion.

To deduce Laplace's theorem, we have

$$u = \psi\{x + hf(u)\}, \text{ therefore } \psi^{-1}(u) - hf(u) = x.$$

Here, therefore, $\psi^{-1}(x) - hf(x)$ must be written for $f(x)$, whence

$$X = x - \psi^{-1}\{\phi(x)\} + hf\{\phi(x)\}.$$

Assume $\phi(x) = \psi(x)$, then $X = hf\{\psi(x)\}$, and substituting in (11),

$$F(u) = F\{\psi(x)\} + hf\{\psi(x)\}\,F'\{\psi(x)\}\,\psi'(x)$$
$$+ \frac{h^2}{1.2}\frac{d}{dx}f\{\psi(x)\}^2\,F'\psi(x)\,\psi'(x) \ldots\ldots (13),$$

which is but one of an infinite number of possible developments.

As a final illustration of the theorem (11), let it be supposed that we are in possession of the solution of the equation $f(u) = x$, which we shall represent by $u = v$, and that we desire to express the solution of the equation

$$f(u) + f_1(u) = x$$

in a series which shall converge rapidly when $f_1(u)$ is small.

Writing in (11) u for $F(u)$, and $f(u) + f_1(u)$ for $f(u)$, we have

$$X = x - f\{\phi(x)\} - f_1\{\phi(x)\}.$$

Let $x - f\{\phi(x)\} = 0$, then $\phi(x) = f^{-1}(x) = v$, and $X = -f_1(v)$; therefore

$$u = v - f_1(v)\frac{dv}{dx} + \frac{1}{1.2}\frac{d}{dx}f_1(v)^2\frac{dv}{dx} - \frac{1}{1.2.3}\frac{d^2}{dx^2}f_1(v)^3\frac{dv}{dx} \dots (14).$$

If $F(u) = u$, we obtain from the general theorem that root of the equation $f(u) = x$, which is represented by $\phi(v)$, where v is the least root of the equation

$$f\{\phi(v)\} = x.$$

Methods of determining the form of ϕ, so as to accomplish particular objects, will suggest themselves to the reader.

Lincoln, Dec. 1847.

ON THE THEORY OF ELIMINATION.

By Arthur Cayley.

Suppose the variables $X_1, X_2, \dots$, g in number, are connected by the (h) linear equations

$$\Theta_1 = \alpha_1 X_1 + \alpha_2 X_2 \dots = 0,$$
$$\Theta_2 = \beta_1 X_1 + \beta_2 X_2 \dots = 0.$$
$$\vdots$$

These equations not being all independent, but connected by the k linear equations

$$\Phi_1 = \alpha_1'\Theta_1 + \alpha_2'\Theta_2 + \dots = 0,$$
$$\Phi_2 = \beta_1'\Theta_1 + \beta_2'\Theta_2 + \dots = 0.$$
$$\vdots$$

These last equations not being independent, but connected by the (l) linear equations

$$\Psi_1 = \alpha_1''\Phi_1 + \alpha_2''\Phi_2 + \ldots . = 0,$$
$$\Psi_2 = \beta_1''\Phi_1 + \beta_2''\Phi_2 + \ldots . = 0,$$
$$\vdots$$

and so on for any number of systems of equations.

Suppose also that $g - h + k - l + \ldots = 0$; in which case the number of quantities $X_1,\ X_2 \ldots$ will be equal to the number of really independent equations connecting them, and we may obtain by the elimination of these quantities a result $\nabla = 0$.

To explain the formation of this final result, write

$$\nabla = \begin{array}{|c|c|c}
\hline
\begin{matrix}\alpha_1\ \beta_1 \ldots \\ \alpha_2\ \beta_2 \\ \vdots\end{matrix} & & \\
\hline
\begin{matrix}\alpha_1'\ \alpha_2' \ldots \\ \beta_1'\ \beta_2' \\ \vdots\end{matrix} & \begin{matrix}\alpha_1''\ \beta_1'' \ldots \\ \alpha_2''\ \beta_2'' \\ \vdots\end{matrix} & \\
\hline
 & \begin{matrix}\alpha_1'''\ \alpha_2''' \ldots \\ \beta_1'''\ \beta_2''' \\ \vdots\end{matrix} & \\
\cline{2-3}
\end{array}$$

which for shortness may be thus represented,

$$\nabla = \begin{array}{|c|c|c}
\hline
\Omega & & \\
\hline
\Omega' & \Omega'' & \\
\hline
 & \Omega''' & \\
\cline{2-3}
\end{array}$$

where $\Omega, \Omega', \Omega'', \Omega''', \Omega''''\ldots$ contain respectively $h, h, l, l, n, n, \ldots$ vertical rows, and $g,\ k,\ k,\ m,\ m,\ p, \ldots$ horizontal rows.

It is obvious from the form in which these systems have been arranged, what is meant by speaking of a certain number of the vertical rows of Ω' and the *supplementary* vertical rows of Ω; or of a certain number of the horizontal rows of Ω'' and the *supplementary* horizontal rows of Ω', &c.

Suppose there is only one system of equations, or $g = h$, we have here only a single system Ω, which contains h vertical and h horizontal rows, and ∇ is simply the determinant formed with the system of quantities Ω. We may write in this case $\nabla = Q$.

Suppose two sets of equations, or $g = h - k$: we have here two systems $\Omega,\ \Omega'$, of which Ω contains h vertical and $h - k$

horizontal rows, Ω' contains h vertical and k horizontal rows. From any k of the h vertical rows of Ω' form a determinant, and call this Q'; from the supplementary $h-k$ vertical rows of Ω form a determinant, and call this Q: then Q' divides Q, and we have $\nabla = Q \div Q'$.

Suppose three sets of equations, or $g = h - k + l$: we have here three systems, Ω, Ω', Ω'', of which Ω contains h vertical and $h - k + l$ horizontal rows, Ω' contains h vertical and k horizontal rows, Ω'' contains l vertical and k horizontal rows. From any l of the k horizontal rows of Ω'' form a determinant, and call this Q''; from the $\overline{k-l}$ supplementary horizontal rows of Ω', choosing the vertical rows at pleasure, form a determinant, and call this Q'; from the $h-k+l$ supplementary vertical rows of Ω form a determinant, and call this Q: then Q'' divides Q', this quotient divides Q, and we have $\nabla = Q \div (Q' \div Q'')$.

Suppose four sets of equations, or $g = h - k + l - m$: we have here four systems, Ω, Ω', Ω'', and Ω''', of which Ω contains h vertical and $h - k + l - m$ horizontal rows, Ω' contains h vertical and k horizontal rows, Ω'' contains l vertical and k horizontal rows, and Ω''' contains l vertical and m horizontal rows. From any m of the l vertical rows of Ω''' form a determinant, and call this Q'''; from the $\overline{l-m}$ supplementary vertical rows of Ω'', choosing the horizontal rows at pleasure, form a determinant, and call this Q''; from the $k - l + m$ supplementary horizontal rows of Ω', choosing the vertical rows at pleasure, form a determinant, and call this Q'; from the $h - k + l - m$ supplementary vertical rows of Ω form a determinant, and call this Q: then Q''' divides Q'', this quotient divides Q'', this quotient divides Q', this quotient divides Q, and $\nabla = Q \div \{Q' \div (Q'' \div Q''')\}$. The mode of proceeding is obvious.

It is clear, that if all the coefficients $\alpha, \beta, \ldots$ be considered of the order unity, ∇ is of the order $h - 2k + 3l -$ &c.

What has preceded constitutes the theory of elimination alluded to in my memoir "On the Theory of Involution in Geometry," *Journal*, vol. II. p. 52-61. And thus the problem of eliminating any number of variables $x, y \ldots$ from the same number of equations $U = 0$, $V = 0, \ldots$ (where $U_1 V_1 \ldots$ are homogeneous functions of any orders whatever) is completely solved; though, as before remarked, I am not in possession of any method of arriving *at once* at the final result in its most simplified form; my process, on the contrary, leads me to a result encumbered by an extraneous factor, which is only got rid of by a number of successive

divisions less by two than the number of variables to be eliminated.

To illustrate the preceding method, consider the three equations of the second order,

$$U = ax^2 + by^2 + cz^2 + lyz + mzx + nxy = 0,$$
$$V = a'x^2 + b'y^2 + c'z^2 + l'yz + m'zx + n'xy = 0,$$
$$W = a''x^2 + b''y^2 + c''z^2 + l''yz + m''zx + n''xy = 0.$$

Here, to eliminate the fifteen quantities x^4, y^4, z^4, y^3z, z^3x, x^3y, yz^3, zx^3, xy^3, y^2z^2, z^2x^2, x^2y^2, x^2yz, y^2zx, z^2xy, we have the eighteen equations

$$x^2U = 0,\quad y^2U = 0,\quad z^2U = 0,\quad yzU = 0,\quad zxU = 0,\quad xyU = 0,$$
$$x^2V = 0,\quad y^2V = 0,\quad z^2V = 0,\quad yzV = 0,\quad zxV = 0,\quad xyV = 0,$$
$$x^2W = 0,\quad y^2W = 0,\quad z^2W = 0,\quad yzW = 0,\quad zxW = 0,\quad xyW = 0,$$

equations, however, which are not independent, but are connected by

$$a''x^2V + b''y^2V + c''z^2W + l''yzV + m''zxV + n''xyV$$
$$-(a'x^2W + b'y^2W + c'z^2W + l'yzW + m'zxW + n'xyW) = 0,$$
$$ax^2W + by^2W + cz^2W + lyzW + mzxW + nxyW$$
$$-(a''x^2U + b''y^2U + c''z^2U + l''yzU + m''zxU + n''xyU) = 0,$$
$$a'x^2U + b'y^2U + c'z^2U + l'yzU + m'zxU + n'xyU$$
$$-(ax^2V + by^2V + cz^2V + lyzV + mzxV + nxyV) = 0.$$

And arranging these coefficients in the required form, we have the following value of ∇.

a						a'						a''					
	b						b'						b''				
		c						c'						c''			
	l		b				l'		b'				l''		b''		
		m		c				m'		c'				m''		c''	
n					a	n'					a'	n''					a''
		l	c					l'	c'					l''	c''		
m				a		m'				a'		m''				a''	
	n				b		n'				b'		n''				b''
	c	b	l				c'	b'	l'				c''	b''	l''		
c		a		m		c'		a'		m'		c''		a''		m''	
b	a				n	b'	a'				n'	b''	a''				n''
l			a	n	m	l'			a'	n'	m'	l''			a''	n''	m''
	m		n	b	l		m'		n'	b'	l'		m''		n''	b''	l''
		n	m	l	c			n'	m'	l'	c'			n''	m''	l''	c''
						a''	b''	c''	l''	m''	n''	$-a'$	$-b'$	$-c'$	$-l'$	$-m'$	$-n'$
$-a''$	$-b''$	$-c''$	$-l''$	$-m''$	$-n''$							a	b	c	l	m	n
a'	b'	c'	l'	m'	n'	$-a$	$-b$	$-c$	$-l$	$-m$	$-n$						

which may be represented as before by

$$\nabla = \boxed{\frac{\Omega}{\Omega'}}\,.$$

Thus, for instance, selecting the first, second, and sixth lines of Ω' to form the determinant Q', we have $Q' = a''(a'b'' - a''b')$; and then Q must be formed from the third, fourth, fifth, seventh, &c. . . . . . eighteenth lines of Ω. (It is obvious that if Q' had been formed from the first, second, and third lines of Ω', we should have had $Q' = 0$. The corresponding value of Q would also have vanished, and an illusory result be obtained; and similarly for several other combinations of lines.)

58, *Chancery Lane*, *Nov.* 4, 1847.

ON THE EXPANSION OF INTEGRAL FUNCTIONS IN A SERIES OF LAPLACE'S COEFFICIENTS.

By Arthur Cayley.

Suppose

$$\begin{aligned} S &= A\mu^s + A_1\mu^{s-1} + \dots \\ &= aQ_s + a_1 Q_{s-1} + \dots \quad \dots\dots\dots\dots (1), \end{aligned}$$

where, as usual, Q_0, Q_1, &c. are the coefficients of the successive powers of p in $(1 - 2\mu p + p^2)^{-\frac{1}{2}}$. Assume

$$S = \frac{\sqrt{(1 + \Delta^2)}}{\sqrt{(1 - 2\mu\Delta + \Delta^2)}}\, C \quad \dots\dots\dots\dots (2),$$

where Δ refers to C; then expanding this expression, first in powers of μ, and then in a series of terms of the form $\sqrt{(1 + \Delta^2)}.Q$, and comparing these with the preceding values of S,

$$A_{s-q} = \frac{1.3 \dots 2q - 1}{2.4 \dots\ 2q}\left(\frac{2\Delta}{1 + \Delta^2}\right)^q . C,$$

$$a_{s-r} = \sqrt{(1 + \Delta^2)}\,.\,\Delta^r\,.\,C \quad \dots\dots\dots\dots (3).$$

Assume $\dfrac{2\Delta}{1 + \Delta^2} = \delta$, that is,

$$\Delta = \frac{1}{\delta}\left\{1 - \sqrt{(1 - \delta)^2}\right\},\quad \sqrt{(1 + \Delta)^2} = \frac{\sqrt{2}}{\delta}\left\{1 - \sqrt{(1 - \delta)^2}\right\}^{\frac{1}{2}} \dots (4).$$

Then $$\delta^q C = \frac{2.4 \ldots 2q}{1.3 \ldots 2q-1} A_{s-q},$$

$$a_{s-r} = \frac{\sqrt{2}}{\delta^{r+1}} \{1 - \sqrt{(1-\delta)^2}\}^{r+\frac{1}{2}} . C \ldots\ldots\ldots (5);$$

or, expanding this last equation in powers of δ,

$$a_{s-r} = \frac{(2r+1)}{2^r} \delta^r \left(\frac{1}{2r+1} + \frac{1}{2} . \frac{\delta^2}{2^2} + \frac{(2r+7)}{2.4} \frac{\delta^4}{2^4} + \frac{(2r+9)(2r+11)}{2.4.6} \frac{\delta^4}{2^6} + \ldots \right) C \ldots . (6);$$

or, replacing the successive terms of the form $\delta^q . C$ by their respective values,

$$a_{s-r} = (2r+1) \left\{ \frac{2.4\ldots 2r}{3.5\ldots(2r+1)\, 2^r} A_{s-r} + \frac{4.6\ldots(2r+4)}{3.5\ldots(2r+3)\, 2^{r+2}} A_{s-r-2} \ldots + \frac{(2k+2)(2k+4)\ldots(2r+4k)}{3.5 \ldots (2r+2k+1)\, 2^{r+2k}} A_{s-r-2k} \ldots \right\} \ldots (7).$$

Thus, if $S = \mu^s$, so that $A_{s-r} = 0$, except in the particular case $A = 1$,

$$a_{2k-1} = 0,$$

$$a_{2k} = \frac{2s - 4k + 1}{2s} \, \frac{(2k+2)(2k+4) \ldots 2s}{3.5 \ldots (2s-2k+1)} \ldots (8),$$

or $$\mu^s = \frac{1}{2^s} \Sigma \left\{ (2s - 4k + 1) \frac{(2k+2)(2k+4) \ldots 2s}{3.5 \ldots (2s-2k+1)} Q_{s-2k} \right\} \ldots (9),$$

which of course includes the preceding case. By substituting the expanded values of the coefficients Q, or again, by determining the value of $(1-\mu)^s$ in terms of these coefficients, and equating it with that given in Murphy's *Electricity*, p. 10, or in a variety of other ways, a series of identical equations involving sums of factorials may readily be obtained. The mode of employing the general theory of the separation of symbols made use of in the preceding example, may easily be applied to the solution of analogous questions.

NOTES ON HYDRODYNAMICS.

III.—*On the Dynamical Equations.*

By G. G. Stokes.

In reducing to calculation the motion of a system of rigid bodies, or of material points, there are two sorts of equations

with which we are concerned; the one expressing the geometrical connexions of the bodies or particles with one another, or with curves or surfaces external to the system, the other expressing the relations between the changes of motion which take place in the system and the forces producing such changes. The equations belonging to these two classes may be called respectively the geometrical, and the dynamical equations. Precisely the same remarks apply to the motion of fluids. The geometrical equations which occur in Hydrodynamics have been already considered by Professor Thomson, in Notes I. and II. The object of the present Note is to form the dynamical equations.

The fundamental hypothesis of Hydrostatics is, that the mutual pressure of two contiguous portions of a fluid, separated by an imaginary plane, is normal to the surface of separation. This hypothesis forms in fact the mathematical definition of a fluid. The equality of pressure in all directions is in reality not an independent hypothesis, but a necessary consequence of the former. A proof of this may be seen at the commencement of Prof. Miller's *Hydrostatics*. The truth of our fundamental hypothesis, or at least its extreme nearness to the truth, is fully established by experiment. Some of the nicest processes in Physics depend upon it; for example, the determination of specific gravities, the use of the level, the determination of the zenith by reflection from the surface of mercury.

The same hypothesis is usually made in Hydrodynamics. If it be assumed, the equality of pressure in all directions will follow as a necessary consequence. This may be proved nearly as before, the only difference being that now we have to take into account, along with the impressed forces, forces equal and opposite to the effective forces. The verification of our hypothesis is however much more difficult in the case of motion, partly on account of the mathematical difficulties of the subject, partly because the experiments do not usually admit of great accuracy. Still, theory and experiment have been in certain cases sufficiently compared to shew that our hypothesis may be employed with very little error in many important instances. There are however many phenomena which point out the existence of a tangential force in fluids in motion, analogous in some respects to friction in the case of solids, but differing from it in this respect, that whereas in solids friction is exerted at the surface, and between points which move relatively to each other with a finite velocity, in fluids friction is exerted throughout the mass, where the

velocity varies continuously from one point to another. Of course it is the same thing to say that in such cases there is a tangential force along with a normal pressure, as to say that the mutual pressure of two adjacent elements of a fluid is no longer normal to their common surface.

The subsidence of the motion in a cup of tea which has been stirred may be mentioned as a familiar instance of friction, or, which is the same, of a deviation from the law of normal pressure; and the absolute regularity of the surface when it comes to rest, whatever may have been the nature of the previous disturbance, may be considered as a proof that all tangential force vanishes when the motion ceases.

It does not fall in with the object of this Note to enter into the theory of the friction of fluids in motion,* and accordingly the hypothesis of normal pressure will be adopted. The usual notation will be employed, as in the preceding Notes. Consider the elementary parallelepiped of fluid comprised between planes parallel to the coordinate planes and passing through the points whose coordinates are x, y, z, and $x + dx,\ y + dy,\ z + dz$. Let X, Y, Z be the accelerating forces acting on the fluid at the point (x, y, z); then, ρ and X being ultimately constant throughout the element, the moving force parallel to x arising from the accelerating forces which act on the element will be ultimately $\rho X\,dx\,dy\,dz$. The difference between the pressures, referred to a unit of surface, at opposite points of the faces $dy\,dz$ is ultimately $\dfrac{dp}{dx}\,dx$, acting in the direction of x negative, and therefore the difference of the total pressures on these faces is ultimately $\dfrac{dp}{dx}\,dx\,dy\,dz$; and the pressures on the other faces act in a direction perpendicular to the axis of x. The effective moving force parallel to x is ultimately $\rho\,\dfrac{D^2x}{Dt^2}\,dx\,dy\,dz$, where, in order to prevent confusion, D is used to denote differentiation when the independent variables are supposed to be t, and three parameters which distinguish one particle of the fluid from another, as for instance the initial coordi-

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* The reader who feels an interest in the subject may consult a memoir by Navier, *Mémoires de l' Académie*, tom. VI. p. 389; another by Poisson, *Journal de l' Ecole Polytechnique*, Cahier XX. p. 139; an abstract of a memoir by M. de Saint-Venant, *Comptes Rendus*, tom. XVII. (Nov. 1843) p. 1240; and a paper in the *Cambridge Philosophical Transactions*, vol. VIII. p. 287.

nates of the particle, while d is reserved to denote differentiation when the independent variables are x, y, z, t. We have therefore, ultimately,

$$\rho \frac{D^2x}{Dt^2} dx\, dy\, dz = \left(\rho X - \frac{dp}{dx}\right) dx\, dy\, dz,$$

with similar equations for y and z. Dividing by $\rho\, dx\, dy\, dz$, transposing, and taking the limit, we get

$$\frac{1}{\rho}\frac{dp}{dx} = X - \frac{D^2x}{Dt^2},\quad \frac{1}{\rho}\frac{dp}{dy} = Y - \frac{D^2y}{Dt^2},\quad \frac{1}{\rho}\frac{dp}{dz} = Z - \frac{D^2z}{Dt^2} \quad \text{.....(1)}.$$

These are the dynamical equations which must be satisfied at every point in the interior of the fluid mass; but they are not at present in a convenient shape, inasmuch as they contain differential coefficients taken on two different suppositions. It will be convenient to express them in terms of differential coefficients taken on the second supposition, that is, that x, y, z, t are the independent variables. Now $\frac{Dx}{Dt} = u$, and on the second supposition u is a function of t, x, y, z, each of which is a function of t on the first supposition. We have, therefore, by Differential Calculus,

$$\frac{Du}{Dt} \quad \text{or} \quad \frac{D^2x}{Dt^2} = \frac{du}{dt} + \frac{du}{dx}\frac{Dx}{Dt} + \frac{du}{dy}\frac{Dy}{Dt} + \frac{du}{dz}\frac{Dz}{Dt};$$

or, since by the definitions of u, v, w,

$$\frac{Dx}{Dt} = u,\quad \frac{Dy}{Dt} = v,\quad \frac{Dz}{Dt} = w,$$

we have
$$\frac{D^2x}{Dt^2} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz},$$

with similar equations for y and z.

Substituting in (1), we have

$$\left.\begin{aligned} \frac{1}{\rho}\frac{dp}{dx} &= \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}, \\ \frac{1}{\rho}\frac{dp}{dy} &= \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}, \\ \frac{1}{\rho}\frac{dp}{dz} &= \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}, \end{aligned}\right\} \quad \text{......(2)},$$

which is the usual form of the equations.

The equations (1) or (2), which are physically considered the same, determine completely, so far as Dynamics alone are concerned, the motion of each particle of the fluid. Hence any other purely dynamical equation which we might set down would be identically satisfied by (1) or (2). Thus, if we were to consider the fluid which at the time t is contained within a closed surface S, and set down the last three equations of equilibrium of a rigid body between the pressures exerted on S, the moving forces due to the accelerating forces acting on the contained fluid, and the effective moving forces reversed, we should not thereby obtain any new equation. The surface S may be either finite or infinitesimal, as, for example, the surface of the elementary parallelepiped with which we started. Thus we should fall into error if we were to set down these three equations for the parallelepiped, and think that we had thereby obtained three new independent equations.

If the fluid considered be homogeneous and incompressible, ρ is a constant. If it be heterogeneous and incompressible, ρ is a function of x, y, z, t, and we have the additional equation $\dfrac{D\rho}{Dt} = 0$, or

$$\frac{d\rho}{dt} + u\frac{d\rho}{dx} + v\frac{d\rho}{dy} + w\frac{d\rho}{dz} = 0 \quad\ldots\ldots\ldots(3),$$

which expresses the fact of the incompressibility. If the fluid be elastic and homogeneous, and at the same temperature θ throughout, and if moreover the change of temperature due to condensation and rarefaction be neglected, we shall have

$$p = k\rho\,(1 + \alpha\theta) \quad\ldots\ldots\ldots\ldots\ldots(4),$$

where k is a given constant, depending on the nature of the gas, and α a known constant which is the same for all gases. The numerical value of α, as determined by experiment, is .00366, θ being supposed to refer to the centigrade thermometer.

If the condensations and rarefactions of the fluid be rapid, we may without inconsistency take account of the increase of temperature produced by compression, while we neglect the communication of heat from one part of the mass to another. The only important problem coming under this class is that of sound. If we suppose the changes in pressure and density small, and neglect the squares of small quantities, we have, putting p_1, ρ_1 for the values of p, ρ in equilibrium,

$$\frac{p - p_1}{p_1} = K\frac{\rho - \rho_1}{\rho_1} \quad\ldots\ldots\ldots\ldots\ (5),$$

K being a constant which, as is well known, expresses the ratio of the specific heat of the gas considered under a constant pressure to its specific heat when the volume is constant. We are not, however, obliged to consider specific heat at all; but we may if we please regard K merely as the value of $\dfrac{d \log p}{d \log \rho}$ for $\rho = \rho_1$, p being that function of ρ which it is in the case of a mass of air suddenly compressed or dilated. In whichever point of view we regard K, the observation of the velocity of sound forms the best mode of determining its numerical value.

It will be observed that in the proof given of equations (1) it has been supposed that the pressure exerted by the fluid outside the parallelepiped was exerted wholly on the fluid forming the parallelepiped, and not partly on this portion of fluid and partly on the fluid at the other side of the parallelepiped. Now, the pressure arising directly from molecular forces, this imposes a restriction on the diminution of the parallelepiped, namely that its edges shall not become less than the radius of the sphere of activity of the molecular forces. Consequently we cannot, mathematically speaking, suppose the parallelepiped to be indefinitely diminished. It is known, however, that the molecular forces are insensible at sensible distances, so that we may suppose the parallelepiped to become so small that the values of the forces, &c., for any point of it, do not sensibly differ from their values for one of the corners, and that all summations with respect to such elements may be replaced without sensible error by integrations; so that the values of the several unknown quantities obtained from our equations by differentiation, integration, &c. are sensibly correct, so far as this cause of error is concerned; and that is all that we can ever attain to in the mathematical expression of physical laws. The same remarks apply as to the bearing on our reasoning of the supposition of the existence of ultimate molecules, a question into which we are not in the least called upon to enter.

There remains yet to be considered what may be called the dynamical equation of the bounding surface.

Consider, first, the case of a fluid in contact with the surface of a solid, which may be either at rest or in motion. Let P be a point in the surface, about which the curvature is not infinitely great, ω an element of the surface about P, PN a normal at P, directed into the fluid, and let $PN = h$. Through N draw a plane A perpendicular to PN, and pro-

ject ω on this plane by a circumscribing cylindrical surface. Suppose h greater than the radius r of the sphere of activity of the molecular forces, and likewise large enough to allow the plane A not to cut the perimeter of ω. For the reason already mentioned r will be neglected, and therefore no restriction imposed on h on the first account. Let Π be the pressure sustained by the solid, referred to a unit of surface, Π having the value belonging to the point P, and let p' be the pressure of the fluid at N. Consider the element of fluid comprised between ω, its projection on the plane A, and the projecting cylindrical surface. The forces acting on this element are, first, the pressure of the fluid on the base, which acts in the direction NP, and is ultimately equal to $p'\omega$; secondly, the pressure of the solid, which ultimately acts along PN and is equal to $\Pi\omega$; thirdly, the pressure of the fluid on the cylindrical surface, which acts everywhere in a direction perpendicular to PN; and, lastly, the moving forces due to the accelerating forces acting on the fluid; and this whole system of forces is in equilibrium with forces equal and opposite to the effective moving forces. Now the moving forces due to the accelerating forces acting on the fluid, and the effective moving forces, are both of the order ωh, and therefore, whatever may be their directions, vanish in the limit compared with the force $p'\omega$, if we suppose, as we may, that h vanishes in the limit. Hence we get from the equation of the forces parallel to PN, passing to the limit,

$$p = \Pi \dots\dots\dots\dots\dots\dots (6),$$

p being the limiting value of p', or the result obtained by substituting in the general expression for the pressure the coordinates of the point P for x, y, z.

It should be observed that, in proving this equation, the forces on which capillary phenomena depend have not been taken into account. And in fact it is only when such forces are neglected that equation (6) is true.

In the case of a liquid with a free surface, or more generally in the case of two fluids in contact, it may be proved, just as before, that equation (6) holds good at any point in the surface, p, Π being the results obtained on substituting the coordinates of the point considered for the general coordinates in the general expressions for the pressure in the two fluids respectively. In this case, as before, capillary attraction is supposed to be neglected.

NOTE ON THE AXIS OF INSTANTANEOUS ROTATION.

By G. G. Stokes.

The most general instantaneous motion of a rigid body moveable in all directions about a fixed point consists in a motion of rotation about an axis passing through that point. This elementary proposition is sometimes assumed as self-evident, sometimes deduced as the result of an analytical process. It ought hardly, perhaps, to be assumed; but it does not seem desirable to refer to a long algebraical process, for the demonstration of a theorem so simple. Yet I am not aware of a geometrical proof anywhere published, which might be referred to. The object of this Note is to supply the deficiency.

In what follows, by the expression *velocity of a line of particles* passing through the fixed point, is to be understood the angular velocity of a line fixed in the body, measured as if the conical surface described by the line were developed.

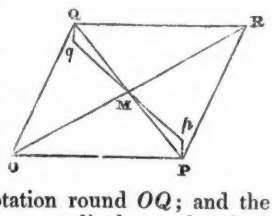
Let O be the fixed point, P any plane, fixed in space, passing through O, $A'OA$ any line of particles in the plane P. Then, if OA have a velocity to either side of the plane P, the produced part OA' must have a velocity to the opposite side. Therefore, as we pass in the plane P from OA continuously round to OA', the velocity of the line of particles considered passes from one side of the plane to the other; therefore for some intermediate line OB there is no velocity to either side of the plane. If OA have no velocity to either side of the plane, OA may be taken for OB. If now the velocity of OB be not zero, through OB draw a plane P', fixed in space, perpendicular to the plane P. Then, as before proved, there exists in the plane P' a line of particles OI, which has no velocity to either side of that plane. But this being the case, it is easy to see that OI is for the instant at rest; for if not, OI has a velocity in the plane P', which we are at liberty to suppose produced by the body's revolving round a normal to that plane, and this gives OB a velocity perpendicular to the plane P; and since the only remaining displacement of the body which is possible, namely a rotation round OI, gives OB a velocity in the plane P, the whole velocity of OB must take place to one side of that plane, which is contrary to hypothesis. If the velocity of OB be zero OB may be taken for OI. Hence the existence of at least one line of particles OI, which is for the instant at rest, is established; but it is evident that, unless the whole body be for the instant at rest, there cannot be

more than one such line, if OI and its produced part be counted as one. Also, since OI is for the instant at rest, it is evident that if the body be not at rest it must for the instant be turning round OI. The conception of the measure of its angular velocity about OI presents no difficulty.

Let ω be this angular velocity, so that the displacement of the body in the time dt is ultimately the same as if the body were turned through an angle ωdt about OI, which remained fixed; and let ω', ω''....be any number of quantities such that if the body were turned round the lines OI', OI''....in succession through the angles $\omega' dt$, $\omega'' dt$.... its total displacement would ultimately be the same as its actual displacement in the time dt. It is evident, (1) that the order of the displacements about OI', OI''.... is immaterial in the limit, (2) that the quantities ω', ω''...., considered separately, represent angular velocities about the lines OI', OI''...., (3) that it is immaterial whether these lines be regarded as fixed in space or fixed in the body. The angular velocity ω is defined to be the resultant of the angular velocities ω', ω''.... It is easily seen that, according to this definition, the resultant angular velocity is the same as the resultant of the partial resultants obtained by arranging the angular velocities into groups and finding the resultant of each group.

If two angular velocities about axes passing through a fixed point be represented in magnitude and direction by two adjacent sides of a parallelogram, their resultant will be represented by that diagonal of the parallelogram which passes between those sides. Of course regard must be had to the direction of rotation, so as always to draw the line towards that pole of the axis of rotation which is considered positive. This proposition may be proved as in Airy's Tract on Precession and Nutation (Introd.), or as follows.

Let OP, OQ represent the angular velocities ω', ω'', and let ω be their resultant. Suppose OP, OQ in the plane of the paper, and let that pole of an axis of rotation which corresponds to the south pole of the earth be reckoned positive. Then in the time dt, Q is carried downwards to q by the rotation round OP, and P is carried upwards to p by the rotation round OQ; and the right lines Qq, Pp are ultimately perpendicular to the plane of the paper. Now Qq, Pp are as the angular velocities ω', ω'', multiplied respectively by the perpendiculars let fall



from Q, P on OP, OQ. But ω', ω'' are as OP, OQ, and the perpendiculars are as OQ, OP, and therefore Qq, Pp are equal. Join PQ, pq; these lines will ultimately intersect in their middle point M, because we have ultimately the isosceles triangles MPp, MQq, which are equal in every respect; and therefore, as the line of particles PQ passes into the new position pq, the point M will be ultimately at rest, so that the instantaneous axis passes through M, and therefore through R, if $OPRQ$ be the parallelogram constructed on OP, OQ. Hence the diagonal OR represents the resultant of ω', ω'' in direction.

Now Qq may be conceived as described, either by the rotation $\omega' dt$ round OP, or by the rotation $\omega\, dt$ round OR. Hence ω, ω' are inversely as the perpendiculars let fall from Q on OR, OP, or directly as the sines of QOP, QOR, or as OR, OP; and therefore OR represents the resultant of ω', ω'' in magnitude as well as direction.

This proposition having been established, we may compound and resolve angular velocities about axes passing through a fixed point just like forces whose directions pass through one point. Thus the most general instantaneous motion of a rigid body moveable about a fixed point O may be defined, either by an arbitrary angular velocity about an arbitrary axis passing through O, or by three arbitrary angular velocities about three given axes, rectangular or oblique, passing through O. In the case of rectangular axes, which alone need be considered, the linear velocities of the particles may be expressed without the least difficulty in terms of the angular velocities about the axes.

Perhaps the proposition enunciated at the beginning of this note might have been proved rather more simply by commencing with the composition of angular velocities; but this method appears more synthetical than the one adopted.

REMARK ON THE THEORY OF HOMOGENEOUS ELASTIC SOLIDS.

By G. G. Stokes.

In a paper on Elastic Solids, published in the last Number of this Journal, Professor Thomson speaks of my paper on the same subject as being the only work in which the equations of equilibrium or motion of a homogeneous elastic solid are given with the two arbitrary independent constants which they must contain. This is so far correct that there is no

other work, of which I am aware, in which these equations are *insisted on* as being those which it is absolutely necessary to adopt; but the equations have been *obtained* by M. Cauchy, and a different method of arriving at them has been pointed out by Poisson.* Poisson seems to have been so fully satisfied with the correctness of the equations with but one constant, which he had obtained from his molecular theory, that he has not even formed the equations with the two constants. M. Cauchy, however, in speaking of the reduction of the two constants to one, and of analogous reductions in the case of crystallized solids, expressly remarks, "Or ce que pourrait faire croire que dans la théorie des corps élastiques il convient d'opérer les diverses réductions dont nous venons de parler, c'est que les expériences faites sur des corps dont l'élasticité reste à peu près la même en tous sens paraissent s'accorder spécialement avec les formules qu'on obtient quand on suppose vérifiée la condition (14)," † the condition, that is, which expresses the relation between the two constants. I have endeavoured in my paper to shew how the equations with one constant might appear consistent with experiment in a variety of cases, and yet lead to enormous errors in the determination of the cubical compressibility of solids. Of course the preceding remarks do not apply to the equations of motion of the luminiferous ether, which it appears must be regarded as an elastic solid, in treating of the dynamics of light. In these equations it seems to be generally allowed that the two arbitrary constants must be retained, or rather that one of them must be considered as infinitely great, the ether being regarded as incompressible.

Pembroke College, Feb. 21, 1848.

ON THE MATHEMATICAL THEORY OF ELECTRICITY IN EQUILIBRIUM.

II.—A STATEMENT OF THE PRINCIPLES ON WHICH THE MATHEMATICAL THEORY OF ELECTRICITY IS FOUNDED.

By William Thomson.

This paper may be regarded as introductory to some others which will follow, containing various investigations in

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* *Journal de l'Ecole Polytechnique*, Cahier xx. p. 82.

† *Exercises*, tom. iv. p. 298.

the Theory of Electricity. The fundamental mathematical principles of the phenomena of Electricity in Equilibrium are stated and explained in as concise a manner as seems consistent with clearness. To avoid lengthening the paper and unnecessarily distracting the attention of the reader, no details are given with reference to the experiments which have been, or which might be, made for establishing the various propositions asserted; and, for the same reasons, scarcely any allusion is made to the history of the subject. With regard to the nature of the evidence on which the mathematical theory of electricity rests, the reader is referred to a paper "On the Elementary Laws of Statical Electricity," (*Cambridge and Dublin Mathematical Journal*, Vol. I. p. 75) where, besides some general explanations on the subject, the works containing accounts of the actual experimental researches of principal importance are indicated. That paper is marked as the first of a series which it was my intention to publish in this *Journal*, and of which the second now appears. In this series it will not be attempted to adhere to a systematic course of investigations such as might constitute a complete treatise on the subject; and my only reason for publishing this introductory article is for the sake of reference in other papers, there being no published work in which the principles are stated in a sufficiently concise and correct form, independently of any hypothesis, to be altogether satisfactory in the present state of science.

The Two Kinds of Electricity.

1. If a piece of glass and a piece of resin are rubbed together and then separated, it is found that they attract one-another mutually. The term *electricity** has been applied to the agency developed in this operation; the excitation of the bodies, to which the attractive force is due, is called *electrical*, and the bodies so excited are said to be *electrified*, or to be *charged with electricity*.

If second pieces of glass and resin be rubbed together and then separated, and placed in the neighbourhood of the first pair of electrified bodies, it may be observed—

(1) *That the two pieces of glass repel one another.*
(2) *That each piece of glass attracts each piece of resin.*
(3) *That the two pieces of resin repel one another.*

Hence it is inferred that the two pieces of glass possess electrical properties which differ in their characteristics from

* From ἤλεκτρον, amber, on account of such phenomena having been first observed with amber as one of the substances rubbed together.

those of the resin; and the two kinds of electricity thus indicated are called vitreous and resinous, after the substances on which they are developed. Bodies may in various ways be made electric; but the characteristics presented are always those of either *vitreous electricity* or *resinous electricity*.

An electrified body exerts no force, whether of attraction or of repulsion, upon any non-electric matter. When in any case bodies not previously electrified are observed to be attracted, or urged in any direction, by an electrical mass, it is because the bodies have become *electrically excited by influence*.

If a small piece of glass and a small piece of resin, which have been electrified by mutual friction, be placed successively in the same position in the neighbourhood of an electrified body, they will be acted upon by equal forces, in the same line, but in contrary directions. Hence the two bodies are said to be *equally charged* with the two kinds of electricity respectively.

Electrical Quantity.

2. The force between two electrified bodies depends, *cæteris paribus*, on the amounts of their charges, or on the quantities of electricity which they possess.

If a small piece of glass and a small piece of resin be electrified by mutual friction to such an extent that, when separated and placed at a unit of distance, they attract one another with a unit of force, the quantity of electricity possessed by the former is said to be unity; the latter possesses what may be called a unit of resinous electricity.

If m bodies, each possessing a unit of vitreous electricity, be incorporated together, the single body thus composed is charged with m units of the same kind of electricity: it is said to possess a quantity of electricity equal to m, or its *electrical mass* is m. A similar definition is applicable with reference to the measurement of resinous electricity.

If two bodies possessing equal quantities of vitreous and resinous electricity be incorporated, the single body thus composed will be found either to be non-electric, or to be in such a state that, without the removal of any electricity of either kind from it, it may, merely by an alteration in the distribution of what it already possesses, be deprived of all electrical symptoms. Thus it appears that a body either vitreously or resinously electrified, may be deprived of its charge merely by supplying it with an equal quantity of the other kind of electricity.

In consequence of this fact, we may establish a complete system of algebraic notation with reference to electrical quantity, whether of vitreous or resinous electricity, by adopting as universal the law that the total quantity of electricity possessed by two bodies, or the quantity possessed by one body made up of two, is equal to the sum of the quantities with which they are separately charged. Thus let m be the quantity of electricity with which a vitreously electrified body is charged, and let m' be the quantity contained by a body equally charged with resinous electricity. We must have

$$m + m' = 0,$$

and therefore m' is equal to $-m$. Now it is usual to regard vitreous electricity as positive; and we must therefore regard the other kind as negative; so that a body possessing m units of resinous electricity is to be considered as charged with a quantity $-m$ of electricity.

The Superposition of Electrical Forces.

3. If a body, electrified in a given invariable manner, be placed in the neighbourhood of any number of electrified bodies, it will experience a force which is the resultant of the forces that would be separately exerted upon it by the different bodies if they were placed in succession in the positions which they actually occupy, without any alteration in their electrical conditions.

This law is true even if any number of the bodies considered be merely different parts of one continuous mass.

Cor. 1. The total mechanical action between two electrified bodies, whether parts of one continuous mass or isolated bodies, is the resultant of the forces due to the mutual actions between all the parts of either body and all the parts of the other, if we conceive the two bodies to be arbitrarily divided each into parts in any manner whatever.

Cor. 2. We may, in any electrical problem, imagine the charge possessed by a body to be divided into two or more parts, each distributed arbitrarily with the sole condition that the sum of the quantities of electricity in any very small space of the body due to the different distributions shall be equal to the given quantity of electricity in that space, according to the actual distribution of electricity in the body; and we may consider the force actually exerted upon any other electrified body as equivalent to the resultant of the forces due to these partial distributions.

The Law of Force between Electrified Bodies.

4. The force between two small electrified bodies varies inversely as the square of the distance between them.

COR. If two small bodies be charged respectively with quantities m and m' of electricity, they will mutually repel with a force equal to

$$\frac{mm'}{\Delta^2};$$

(an action which will be really attractive when m and m' have unlike signs, as would be the case were the bodies dissimilarly electrified). For two units, placed at a distance unity, repel with a force equal to unity, and therefore if placed at a distance Δ, they will repel with a force $\frac{1}{\Delta^2}$; and the expression for the repulsion between m units and m' units is deduced from this, according to the principle of the superposition of forces, by multiplying by mm'.

Definition of the Resultant Electrical Force at a Point.

5. Let a unit of negative electricity be conceived to be concentrated at a point P in the neighbourhood of an electrified body or group of bodies, without producing any alteration in the previously existing electrical distribution. The force exerted upon this electrical point is what we shall throughout understand as *the resultant force at P* due to the electricity of the body or bodies considered.

COR. If R be *the resultant force at P* in any case, then the force actually exerted upon an electrical mass m, concentrated at P, will be equal to $-mR$.

Electrical Equilibrium.

6. When a body held at rest is electrified, and when, being either subject to electrical action from other bodies, or entirely isolated, the distribution of its charge remains permanently unaltered, the electricity upon it is said to be in equilibrium.

Electrical equilibrium may be disturbed in various ways. Thus if a body charged with electricity in equilibrium be touched, or even approached by another electrified body, the equilibrium may be broken, and can only be restored after a different distribution has been effected, by a motion of electricity through the body or along its surface: or if a body be initially electrified in any arbitrary manner, whether

by friction or otherwise, it may be that, as soon as the exciting cause is removed, the electricity will either gradually become altered from its initial distribution, by moving slowly through the body, or will suddenly assume a certain definite distribution.

The laws which regulate the distribution of electricity in equilibrium on bodies in various circumstances have been the subject of most important experimental researches; and, having been established with perfect precision by Coulomb, and placed beyond all doubt by verifications afforded in subsequent experiments, they constitute the foundation of an extremely interesting branch of the Mathematical Theory of Electricity. In connexion with these laws, and before stating them, it will be convenient to explain the nature of the distinction which is drawn between the two great classes of bodies in nature, called conductors of electricity, and non-conductors of electricity.

Non-Conductors of Electricity.

7. A body which affords such a resistance to the transmission of electricity through it, or along its surface, that, if it be once electrified in any way, it retains permanently, without any change of distribution, the charge which it has received, is called a non-conductor of electricity.

No body exists in nature which fulfils strictly the terms of this definition; but glass and resin, besides many other substances, are such that they may, within certain limits and subject to certain restrictions, be considered as non-conductors.

Conductors of Electricity.

8. A very extensive class of bodies in nature, including all the metals, many liquids, &c. are found to possess the property that in all conceivable circumstances of electrical excitation, the resultant force at any point within their substance vanishes. Such bodies are called conductors of electricity, since they are destitute of the property, possessed by non-conductors, of retaining permanently, by a resistance to every change, any distribution of electricity arbitrarily imposed; the only kind of distribution which can exist unchanged for an instant being such as satisfies the condition that the resultant force must vanish in the interior.

It is found by experiment that the electricity of a charged conductor rests entirely on its surface, and that the electrical circumstances are not at all affected by the nature of the

interior, but depend solely upon the form of the external conducting surface. Thus the electrical properties of a solid conductor, of a hollow conducting shell, or of a non-conductor enclosed in an envelope, however thin, (the finest gold leaf, for instance,) are identical, provided the external forms be the same. A hollow conductor never shews symptoms of electricity on its interior surface, unless an electrified body be insulated within it; in which case the interior surface will become electrified by *influence*, or by *induction*, in such a way as to make the total resultant force at any point in the conducting matter vanish, by balancing, for any such point, the force due to the electricity of the insulated body.

It has been frequently assumed that electricity penetrates to a finite depth below the surface of conductors; and, in accordance with certain hypothetical ideas regarding the nature of electricity, the "thickness of the stratum" at different points of the surface of a conductor has been considered as a suitable term with reference to the varying or uniform distribution of electricity over the body. All the conclusion with reference to this delicate subject which can as yet be drawn from experiment, is that the "thickness," if it exist at all, must be less than that of the finest gold leaf; and in the present state of science we must regard it as immeasurably small. It may be conceived that the actual thickness of the excited stratum at the surface of an electrified conductor is of the same order as the space through which the physical properties of the pervading matter change continuously from those of the solids to those which characterize the surrounding air.

Electrical Density at any point of a charged surface.

9. In this, and in all the papers which will follow, instead of the expression "the thickness of the stratum", Coulomb's far more philosophical term, *Electrical Density*, will be employed with reference to the distribution of electricity on the surface of a body; a term which is to be understood strictly in accordance to the following definitions, without involving even the idea of a hypothesis regarding the nature of electricity.

The electrical density of a uniformly charged surface is the quantity of electricity distributed over a unit of surface.

The electrical density at any point of a surface, whether the distribution be uniform or not, is the quotient obtained by dividing the quantity of electricity distributed over an infinitely small element at this point, by the area of the element.

Exclusion of all Non-Conductors except Air.

10. In the present paper, and in some others which will follow, no bodies will be considered except conductors; and the air surrounding them, which will be considered as offering a resistance to the transference of electricity between two detached conductors, but as otherwise destitute of electrical properties. A full development of the mathematical theory, of the internal electrical polarization of solid or liquid non-conductors, subject to the influence of electrified bodies, discovered by Faraday (in his Experimental Researches on the specific inductive capacities of non-conducting media), must be reserved for a later communication.*

Insulated Conductors.

11. A conductor separated from the ground, and touched only by air, is said to be insulated. Insulation may be practically effected by means of solid props of matter, such as glass, shell-lac, or gutta percha†; and if the props be sufficiently thin, it is found that their presence does not in any way alter or affect the electrical circumstances, and that their *resisting* power, as non-conductors of electricity, prevents any alteration in the quantity of electricity possessed by the insulated body; so that, however the distribution may be affected by the influence of surrounding bodies, it is only by a temporary breaking of the insulation that the absolute charge can be increased or diminished.

If an insulated uncharged conductor be placed in the neighbourhood of bodies charged with electricity, it will become "electrified by influence", in such a manner that its resultant electrical force at every internal point shall counterbalance the force due to the exterior charged bodies: but, in accordance with what has been stated in the preceding paragraph, the total quantity of electricity will remain equal to nothing; that is to say, the two kinds of electricity

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* The results of this Theory were explained briefly in a paper entitled "Note sur les Lois Elémentaires de l'Electricité Statique" (published, in 1845, in *Liouville's Journal*), and more fully in the first paper of the present series, on the "Mathematical Theory of Electricity" (Vol. I. p. 75 of this Journal). A similar view of this subject has been taken by Mossotti, whose investigations are published in a paper entitled "*Discussione Analitica sull' Influenza che l' Azione di un Mezzo Dielettrico ha sulla Distributione dell' Elettricità alla Superficie di piu Corpi Elettrici Disseminati in Esso.* (Vol. XXIV. of the *Memorie della Società Italiana delle Scienze Residente in Modena*, dated 1846.)

† It has been recently discovered by Faraday that gutta percha is one of the best insulators among known substances. (*Philosophical Magazine*, March 1848.)

produced upon it by influence will be equal to one another in amount.

Recapitulation of the Fundamental Laws.

12. The laws of electricity in equilibrium in relation with conductors may; if we tacitly take into account such principles as the superposition of electrical forces, and the invariableness of the quantity of electricity on a body, except by addition or subtraction (in the extended algebraic sense of these terms); be considered as fully expressed in the three following propositions.

I. The repulsion between two electrical points is inversely proportional to the square of their distance.

II. Electricity resides at the boundary of a charged conductor.

III. The resultant force at any point in the substance of a conductor, due to all existing electrified bodies, vanishes.

It has been proved by Green that the second of these laws is a mathematical consequence of the first and third; and it has been demonstrated by La Place that the first law may be inferred from the truth, in a certain particular case, of the second and third. The three laws were, however, first announced by Coulomb, as the result of his experimental researches on the subject.

Objects of the Mathematical Theory of Electricity.

13. The varied problems which occur in the mathematical theory of electricity in equilibrium may be divided into the two great classes of Synthetical and Analytical investigations. In problems of the former class, the object is in each case the determination either of a resultant force or of an aggregate electrical mass, according to special data regarding distributions of electricity: in the latter class, inverse problems, such as the determination of the electrical density at each point of the surface of a conductor in any circumstances, according to the laws stated above, are the objects proposed.

It has been proved (by Green and Gauss) that there is a determinate unique solution of every actual analytical problem of the Theory of Electricity in relation with conductors. The demonstration of this with reference to the complete Theory of Electricity (including the action of solid non-conducting media discovered by Faraday), as well as with reference to the Theories of Heat, Magnetism, and Hydrodynamics, may be deduced from two theorems proved in the

last number of this Journal, "Regarding the Solution of certain Partial Differential Equations."

The full investigation of any actual case of electrical equilibrium will generally involve both analytical and synthetical problems; as it may be desirable, besides determining the distribution, to find the resulting electrical force at points not in the interior of any conductor, or to find the total mechanical action due to the attractions or repulsions of the elements of two conductors, or of two portions of one conductor; and besides it is frequently interesting to verify synthetically the solutions obtained for analytical problems.

Actual Progress in the Mathematical Theory of Electricity.

14. In Poisson's valuable memoirs on this subject, the distribution of electricity on two electrified spheres, uninfluenced by other electric matter, is considered; a complete solution of the analytical problem is arrived at; and various special cases of interest are examined in detail with great rigor. In a very elaborate memoir* by Plana, the solution given by Poisson is worked out much more fully, the excessive mathematical difficulties in the way of many actual numerical applications of interest being such as to render a work of this kind extremely important.

The distribution of electricity on an ellipsoid (including the extreme cases of elliptic circular discs, and of a straight rod), and the results of consequent *synthetical* investigations, are well known.

The analytical problem regarding an ellipsoid subject to the influence of given electrical masses, has been solved by M. Liouville, by the aid of a very refined mathematical method suggested by some investigations of M. Lamé with reference to corresponding problems in the Theory of Heat.

Green's Essay on Electricity and his other papers on allied subjects contain, besides the solution of several special problems of interest, most valuable discoveries with reference to the general Theory of Attraction, and open the way to much more extended investigations in the Theory of Electricity than any that have yet been published.

Glasgow College, March 4, 1848.

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* *Turin Academy of Sciences*, Tome VII. Sérié II. published separately in a quarto volume of 333 pages; *Turin*, 1845.

III.—GEOMETRICAL INVESTIGATIONS WITH REFERENCE TO THE DISTRIBUTION OF ELECTRICITY ON SPHERICAL CONDUCTORS.*

By William Thomson.

There is no branch of physical science which affords a surer foundation, or more definite objects for the application of mathematical reasoning, than the theory of electricity. The small amount of attention which this most attractive subject has obtained is no doubt owing to the extreme difficulty of the analysis by which even a very limited progress has as yet been made; and no other circumstance could have kept totally excluded from an elementary course of reading, a subject which, besides its great physical importance, abounds so much in beautiful illustrations of ordinary mechanical principles. This character of difficulty and impracticability is not however inseparable from the mathematical theory of electricity: by very elementary geometrical investigations we may arrive at the solution of a great variety of interesting problems with reference to the distribution of electricity on spherical conductors, including Poisson's celebrated problem of the two spheres, and others which might at first sight be regarded as presenting difficulties of a far higher order. The object of the following paper is to present in as simple a form as possible, some investigations of this kind. The methods followed, being for the most part *synthetical,* were suggested by a knowledge of results

* The investigations given in this paper form the subject of the first part of a series of lectures on the Mathematical Theory of Electricity, given in the University of Glasgow during the present session. They are adaptations of certain methods of proof which first occurred to me as applications of *the principle of electrical images*, made with a view to investigating the solutions of various problems regarding spherical conductors, without the explicit use of the differential or integral calculus. The spirit, if not the notation, of the *differential* calculus must enter into any investigations with reference to Green's theory of the potential, and therefore a more extended view of the subject is reserved for a second part of the course of lectures. A complete exposition of the *principle of electrical images* (of which a short account was read at the late meeting of the British Association at Oxford) has not yet been published; but an outline of it was communicated by me to M. Liouville, in three letters of which extracts are published in the *Journal de Mathématiques*, (1845 and 1847, Vols. x. and xii.) A full and elegant exposition of the method indicated, together with some highly interesting applications to problems in geometry not contemplated by me, are given by M. Liouville himself, in an article written with reference to those letters, and published along with the last of them. I cannot neglect the present opportunity of expressing my thanks for the honour which has thus been conferred upon me by so distinguished a mathematician, as well as for the kind manner in which he received those communications, imperfect as they were, and for the favourable mention made of them in his own valuable memoir.

founded on a less restricted view of the theory of electricity; and it must not be considered either that they constitute the best or the easiest way of advancing towards a *complete* knowledge of the subject, or that they would be suitable as instruments of research in endeavouring to arrive at the solutions of new problems.

Insulated Conducting Sphere subject to no External Influence.

1. We may commence with the simplest possible case, that of a spherical conductor, charged with electricity and insulated in a position removed from all other bodies which could influence the distribution of its charge. In this, as in the other cases which will be considered, the various problems, of the analytical and synthetical classes, alluded to in a previous paper (II. § 13)* will be successively subjects of investigation. Thus let us first determine the density at any point of the surface, and then, after verifying the result by shewing that the laws (II. § 12) are satisfied, let us investigate the resultant force at an external point.

Determination of the Distribution.

2. Let a be the radius of the sphere, and E the amount of the charge.

According to Law II. the whole charge will reside on the surface, and, on account of the symmetry, it must be uniformly distributed. Hence, if ρ be the required density at any point, we have

$$\rho = \frac{E}{4\pi a^2}.$$

Verification of Law III.

3. The well-known theorem that the resultant force due to a uniform spherical shell vanishes for any interior point, constitutes the verification required in this case. This theorem was first given by Newton, and is to be found in the *Principia*; but as his demonstration is the foundation of every synthetical investigation which will be given in this paper, it may not be superfluous to insert it here; and accordingly the passage of the *Principia* in which it occurs, translated literally, is given here.

Newton, First Book, Twelfth Section, Prop. LXX. Theorem XXX.

If the different points of a spherical surface attract equally with forces varying inversely as the squares of the distances,

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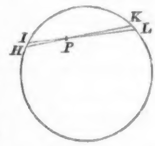
* References in this form will in general be given for the articles in the present series.

a particle placed within the surface is not attracted in any direction.

Let *HIKL* be the spherical surface, and *P* the particle within it. Let two lines *HK, IL,* intercepting very small arcs *HI, KL,* be drawn through *P*; then, on account of the similar triangles *HPI, KPL* (Cor. 3, Lemma VII., Newton), those arcs will be proportional to the distances *HP, LP*; and any small elements of the spherical surface at *HI* and *KL*, each bounded all round by straight lines passing through *P* [and very nearly coinciding with *HK*], will be in the duplicate ratio of those lines. Hence the forces exercised by the matter of these elements on the particle *P* are equal; for they are as the quantities of matter directly, and the squares of the distances, inversely; and these two ratios compounded give that of equality. The attractions therefore, being equal and opposite, destroy one another: and a similar proof shews that all the attractions due to the whole spherical surface are destroyed by contrary attractions. Hence the particle *P* is not urged in any direction by these attractions. Q. E. D.



Digression on the Division of Surfaces into Elements.

4. The division of a spherical surface into infinitely small elements, will frequently occur in the investigations which follow: and Newton's method, described in the preceding demonstration, in which the division is effected in such a manner that all the parts may be taken together in *pairs of opposite elements with reference to an internal point;* besides other methods deduced from it, suitable to the special problems to be examined; will be repeatedly employed. The present digression, in which some definitions and elementary geometrical propositions regarding this subject are laid down, will simplify the subsequent demonstrations, both by enabling us, through the use of convenient terms, to avoid circumlocution, and by affording us convenient means of reference for elementary principles, regarding which repeated explanations might otherwise be necessary.

Explanations and Definitions regarding Cones.

5. If a straight line which constantly passes through a fixed point be moved in any manner, it is said to describe, or generate, a *conical surface* of which the fixed point is the vertex.

If the generating line be carried from a given position continuously through any series of positions, no two of which coincide, till it is brought back to the first, the entire line on the two sides of the fixed point will generate a complete conical surface, consisting of two sheets, which are called *vertical or opposite cones.* Thus the elements HI and KL, described in Newton's demonstration given above, may be considered as being cut from the spherical surface by two *opposite cones* having P for their common vertex.

The Solid Angle of a Cone, or of a complete Conical Surface.

6. If any number of spheres be described from the vertex of a cone as centre, the segments cut from the concentric spherical surfaces will be similar, and their areas will be as the squares of the radii. The quotient obtained by dividing the area of one of these segments by the square of the radius of the spherical surface from which it is cut, is taken as the measure of the *solid angle of the cone.* The segments of the same spherical surfaces made by the opposite cone, are respectively equal and similar to the former. Hence the solid angles of two vertical or opposite cones are equal: either may be taken as the solid angle of the complete conical surface, of which the opposite cones are the two sheets.

Sum of all the Solid Angles round a Point $= 4\pi$.

7. Since the area of a spherical surface is equal to the square of its radius multiplied by 4π, it follows that the sum of the solid angles of all the distinct cones which can be described with a given point as vertex, is equal to 4π.

Sum of the Solid Angles of all the complete Conical Surfaces $= 2\pi$.

8. The solid angles of vertical or opposite cones being equal, we may infer from what precedes that the sum of the solid angles of all the complete conical surfaces which can be described without mutual intersection, with a given point as vertex, is equal to 2π.

Solid Angle subtended at a Point by a terminated Surface.

9. The solid angle subtended at a point by a superficial area of any kind, is the solid angle of the cone generated by a straight line passing through the point, and carried entirely round the boundary of the area.

Orthogonal and Oblique Sections of a small Cone.

10. A very small cone, that is, a cone such that any two positions of the generating line contain but a very small

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angle, is said to be cut at right angles, or orthogonally, by a spherical surface described from its vertex as centre, or by any surface, whether plane or curved, which touches the spherical surface at the part where the cone is cut by it.

A very small cone is said to be cut obliquely, when the section is inclined at any finite angle to an orthogonal section; and this angle of inclination is called the *obliquity of the section.*

The area of an orthogonal section of a very small cone is equal to the area of an oblique section in the same position, multiplied by the cosine of the obliquity.

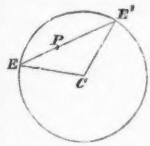
Hence the area of an oblique section of a small cone is equal to the quotient obtained by dividing the product of the square of its distance from the vertex, into the solid angle, by the cosine of the obliquity.

Area of the Segment cut from a Spherical Surface by a small Cone.

11. Let E denote the area of a very small element of a spherical surface at the point E (that is to say, an element every part of which is very near the point E), let ω denote the solid angle subtended by E at any point P, and let PE, produced if necessary, meet the surface again in E': then, a denoting the radius of the spherical surface, we have

$$E = \frac{2a.\omega.PE^2}{EE'}.$$

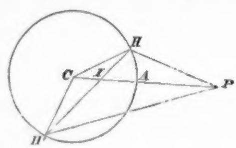
For, the obliquity of the element E, considered as a section of the cone of which P is the vertex and the element E, a section; being the angle between the given spherical surface and another described from P as centre, with PE as radius; is equal to the angle between the radii, EP and EC, of the two spheres. Hence, by considering the isosceles triangle ECE', we find that the cosine of the obliquity is equal to $\frac{\frac{1}{2}EE'}{EC}$ or to $\frac{EE'}{2a}$, and we arrive at the preceding expression for E.



12. *Theorem.** The attraction of a uniform spherical surface on an external point is the same as if the whole mass were collected at the centre.

* This theorem, which is more comprehensive than that of Newton in his first proposition regarding attraction on an external point (Prop. LXXI.),

Let P be the external point, C the centre of the sphere, and CAP a straight line cutting the spherical surface in A. Take I in CP, so that CP, CA, CI may be continual proportionals, and let the whole spherical surface be divided into *pairs of opposite elements with reference to the point I.*



Let H and H' denote the magnitudes of a pair of such elements, situated respectively at the extremities of a chord HH'; and let ω denote the magnitude of the solid angle subtended by either of these elements at the point I.

We have (III. § 11),

$$H = \frac{\omega . IH^2}{\cos CHI}, \quad \text{and } H' = \frac{\omega . IH'^2}{\cos CH'I}.$$

Hence, if ρ denote the density of the surface (II. § 9), the attractions of the two elements H and H' on P are respectively

$$\rho \frac{\omega}{\cos CHI} \cdot \frac{IH^2}{PH^2}, \quad \text{and } \rho \frac{\omega}{\cos CH'I} \cdot \frac{IH'^2}{PH'^2}.$$

Now the two triangles PCH, HCI have a common angle at C, and, since $PC : CH :: CH : CI$, the sides about this angle are proportional. Hence the triangles are similar; so that the angles CPH and CHI are equal, and

$$\frac{IH}{HP} = \frac{CH}{CP} = \frac{a}{CP}.*$$

In the same way it may be proved, by considering the triangles PCH', $H'CI$, that the angles CPH' and $CH'I$ are equal, and that

$$\frac{IH'}{H'P} = \frac{CH'}{CP} = \frac{a}{CP}.$$

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is fully established as a corollary to a subsequent proposition (Prop. LXXIII. Cor. 2). If we had considered the proportion of the forces exerted upon two external points at different distances, instead of, as in the text, investigating the absolute force on one point, and if besides we had taken together all the pairs of elements which would constitute two narrow annular portions of the surface, in planes perpendicular to PC, the theorem and its demonstration would have coincided precisely with Prop. LXXI. of the *Principia.*

* From this we infer that the ratio of IH to HP is constant, whatever be the position of H on the spherical surface, a well-known proposition. (Thomson's *Euclid*, VI. Prop. G.)

Hence the expressions for the attractions of the elements H and H' on P become

$$\rho \frac{\omega}{\cos CHI} \cdot \frac{a^2}{CP^2}, \quad \text{and } \rho \frac{\omega}{\cos CH'I} \cdot \frac{a^2}{CP^2},$$

which are equal, since the triangle HCH' is isosceles; and, for the same reason, the angles CPH, CPH', which have been proved to be respectively equal to the angles CHI, $CH'I$, are equal. We infer that the resultant of the forces due to the two elements is in the direction PC, and is equal to

$$2\omega.\rho. \frac{a^2}{CP^2}.$$

To find the total force on P, we must take the sum of all the forces along PC due to the pairs of opposite elements; and, since the multiplier of ω is the same for each pair, we must add all the values of ω, and we therefore obtain (III. § 8), for the required resultant,

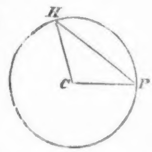
$$\frac{4\pi\rho a^2}{CP^2}.$$

The numerator of this expression; being the product of the density, into the area of the spherical surface; is equal to the mass of the entire charge; and therefore the force on P is the same as if the whole mass were collected at C. Q. E. D.

Cor. The force on an external point, infinitely near the surface, is equal to $4\pi\rho$, and is in the direction of a normal at the point. The force on an internal point, however near the surface, is, by a preceding proposition, equal to nothing.

Repulsion on an element of the Electrified Surface.

13. Let σ be the area of an infinitely small element of the surface at any point P, and at any other point H of the surface let a small element subtending a solid angle ω, at P, be taken. The area of this element will be equal to



$$\frac{\omega.PH^2}{\cos CHP},$$

and therefore the repulsion along HP, which it exerts on the element σ at P, will be equal to

$$\frac{\rho\omega.\rho\sigma}{\cos CHP}, \quad \text{or} \quad \frac{\omega}{\cos CHP}\rho^2\sigma.$$

Now the total repulsion on the element at P is in the direction CP; the component in this direction of the repulsion due to the element H, is

$$\omega \,.\, \rho^2\sigma\,;$$

and, since all the cones corresponding to the different elements of the spherical surface lie on the same side of the tangent plane at P, we deduce, for the resultant repulsion on the element σ,

$$2\pi\rho^2\sigma.$$

From the corollary to the preceding proposition, it follows that this repulsion is half the force which would be exerted on an external point, possessing the same quantity of electricity as the element σ, and placed infinitely near the surface.

[*To be continued.*]

Glasgow College, March 14, 1848.

ON A PRINCIPLE IN THE THEORY OF SURFACES OF THE SECOND ORDER, AND ITS APPLICATION TO M. JACOBI'S METHOD OF GENERATING THE ELLIPSOID.

By R. Townsend.

[*Continued from* p. 108.]

Ever since the vast improvements introduced of late years into modern geometry, some of the ablest mathematicians of Europe have made frequent attempts to find for surfaces of the second order a property analogous to the oldest and most familiarly known property of curves of that order, viz. that the sum or difference, as the case may be, of the distances of any point on the curve from the two foci is always constant: but until last year, when a strikingly analogous property was discovered by Mr. M. Roberts, Fellow of Trinity College, Dublin,*

* In consequence of Mr. Roberts' theorem, when brought down from surfaces to curves of the second order, expressing a property not merely of the curve itself but rather of the infinitely flat surface bounded by that curve, and also in consequence of its not involving an absolute constant but only a variable parameter, even it has been considered by some as furnishing not more at best than a doubtful analogy. To myself, however, the analogy seems remarkably perfect. If in the plane of an ellipse, or of any other curve of the second order, two systems of lines be traced out, expressed respectively by the equations $r \pm r' = 2a$, where a is a variable parameter considered as susceptible of, and of taking in succession all values

their researches were attended with very dubious success, and the inquiry baffled even all the skill and ingenuity of the great Chasles himself, who was inclined to conclude that though the property must exist, it was perhaps impossible in the case of surfaces to express it in geometrical language. Previous to that period the method of generating curves of the second order by the motion of a point, the sum or difference of whose distances from two fixed points was constant, had been put under different forms and viewed in different aspects, and methods more or less analogous had been in consequence proposed for the generation of surfaces of the second order: of these, decidedly the most ingenious and perhaps the most successful was that of M. Jacobi, who communicated it to Professor MacCullagh in (I believe) the year 1842, observing at the same time that it was but a particular case of a more general method, which he had investigated several years before, and of which a sketch had been published in the volume of *Crelle* for the year 1834.*

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from $+$ to $-\infty$, and r and r' are the *rectilinear* distances of the variable points of each line from the two foci of the curve, then will one of these two systems of lines be always one of the two systems of conics confocal with the given curve, while the other will always be the opposite and orthogonal system of confocal conics, and the two systems of lines will always contain between them the curve itself and its two axes. Similarly, if on the surface of an ellipsoid, or of any other surface of the second order, two systems of lines be traced out, expressed respectively by the equations $\rho \pm \rho' = 2a$, where a, as before, is a variable parameter susceptible of and considered as taking in succession all values from $+$ to $-\infty$, and when ρ and ρ' are the *geodesic* distances of the variable points of each line from either pair of the umbilici of the surface, then will one of these two systems of lines be always one of the two systems of lines of curvature of the given surface, while the other will always be the opposite and orthogonal system of lines of curvature, and the two systems of lines will always contain between them the three principal sections of the surface itself. Such was the theorem discovered by Mr. M. Roberts, a theorem which, whether considered as completely or but partially supplying the desired analogy, is unquestionably of surpassing elegance, and which, moreover, if considered in reference to the descriptive construction of the lines of curvature on any given surface of the second order, is undoubtedly of the first practical importance, as furnishing us with a most obvious and simple mechanical method, absolutely identical with the analogous method for plane and for spherical conics, of tracing out by continued motion those beautiful lines on the surface of the ellipsoid.

* The analogy proposed by Professor MacCullagh himself was also remarkable, stating the property of curves as follows. The diameter conjugate to any point assumed arbitrarily on any central curve of the second order intercepts on the two focal radii diverging from that point, two portions which are always equal to each other and to the primary semiaxis of the curve. He shewed that analogously in surfaces—The diametral plane conjugate to any point assumed arbitrarily on any central surface of the second order intercepts on the four bifocal lines diverging from that point, four portions which are always equal to each other and to the primary semiaxis of the surface. In curves, however, the remaining por-

Two fixed points being taken in a right line and two others in a plane, the locus of a moveable point in the plane whose distances from the two fixed points are always equal respectively to the distances of the variable point in the right line from its two fixed points, will be always a curve of the second order.* Such was the aspect in which he viewed the well-known method of generating curves of the second order.

Similarly, three fixed points being taken in a plane and three others in space, the locus of a moveable point in space whose distances from the three fixed points are always equal respectively to the distances of a variable point in the plane from its three fixed points, will be always a surface of the second order.† Such was the method of generating surfaces

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tions of the focal radii destroy each other by addition or subtraction; but in surfaces the same is not the case for the bifocal lines,—in this respect the analogy fails.

* It will be an ellipse or an hyperbola, obviously, according as the distance between the two fixed points in the right line exceeds or is less than the distance between the two fixed points in the plane, the foci of the curve; for in every plane triangle the sum of any two sides must always exceed, while their difference must always be less than the third: in the particular case where the two distances are equal, the locus will obviously be either portion indifferently of the indefinite right line passing through and bounded by the two fixed points in its plane; and so it ought, such being the limiting and transition state between an ellipse and an hyperbola of which the two foci are given.

† The equation of this locus has been exhibited under a very elegant form by Mr. Salmon, who obtained it very readily from the following consideration. Let a, b, c be the three sides of any triangle, and l, m, n the three distances of any point in its plane from the opposite angles respectively; then, since, whatever be the position of the variable point, the three angles mn, nl, lm always make up four right angles, these three angles will be always connected by the following well-known trigonometrical equation,

$$1 - \cos^2 mn - \cos^2 nl - \cos^2 lm + 2 . \cos mn . \cos nl . \cos lm = 0 \,..\, (1);$$

and in this, substituting for the three cosines their values in terms of the sides of the three triangles amn, bnl, clm, and reducing, we get for the general equation which, independent of the position of the variable point, connects the three lines l, m, n with the three sides of the triangle a, b, c, the following:

$$a^2 (l^2 - m^2)(l^2 - n^2) + b^2 (m^2 - n^2)(m^2 - l^2) + c^2 (n^2 - l^2)(n - m^2)$$
$$+ a^2 (b^2 + c^2 - a^2) l^2 + b^2 (c^2 + a^2 - b^2) m^2 + c^2 (a^2 + b^2 - c^2) n^2 - a^2b^2c^2 = 0 \,..\, (2).$$

In which, considering a, b, c as the three sides of the triangle formed by connecting the three fixed points in the plane, we have only to substitute for l^2, m^2, n^2 the squares of the three distances of the moveable point in space from the other three fixed points in order to have the equation of the locus sought, an equation which will be obviously always of the second order, inasmuch as the squares of the variables evidently destroy each other in all the factors of the first three terms.

With respect to the equation (2) Mr. Salmon has also very ingeniously observed that, by taking the variable point outside the figure of the triangle a, b, c, it expresses the general relation connecting the four sides and the two diagonals of every plane quadrilateral.

of the second order which he proposed as analogous to the same well-known method of generating curves of the second order.

It is obvious, *a priori*, that the curve in the former case, whatever be its nature, will be always symmetrical on both sides of the right line passing through the two fixed points in the plane, and that the surface in the latter case, whatever be its nature, will also be always symmetrical on both sides of the plane passing through the three fixed points in space; hence that line in the former case will be always an axis of the curve, and that plane in the latter a principal plane of the surface. Following the analogy of the two modes of generation already considered, the modular and the umbilicar, the three fixed points in space which occupy such an important position in the present mode of generating surfaces of the second order may for the present be also called *foci*, and the triangle formed by connecting them, the *focal triangle;* while, for the sake of distinction, the other triangle, formed by joining the three fixed points in the plane, may, in pursuance of the same analogy, be called the *dirigent* or *directive* triangle. Some names distinctive of these two different triangles being for the present absolutely necessary for the sake of convenience and distinction, the particular names which we have ventured to propose for that purpose are obviously suggested by the analogy of the two methods already considered, and they serve that particular end as well as any others that might easily be suggested in preference; but we hope presently to be able to shew that in relation to the resulting surface they are far from being inapplicable, and that they serve to indicate each the peculiar and distinguishing properties of the triangles to which they are respectively applied.

We now proceed to establish an additional point of analogy between this method of generating surfaces of the second order and the corresponding method of generating curves of that order; viz. that, as in the latter case the two fixed points in the plane are always the foci of the generated curve in the axis of that curve in which they both lie, so in the former case, the three fixed points in space, that is the three angular points of the focal triangle, are always upon the focal conic of the generated surface in the principal plane of that surface in which they all three lie. The most obvious method of proceeding to ascertain whether this were the case or not would be, having investigated the equation of the surface to any coordinate axes assumed arbitrarily, two

in that principal plane and the third perpendicular to it, to transfer the coordinate origin and axes to the centre and axes of the surfaces, and then, having deduced from the resulting equation that of the focal conic in the same principal plane, to try whether the coordinates of the points would satisfy that equation. But this method would in the present instance be exceedingly troublesome, from the great length of the coefficients and the consequent complexity of the necessary algebraical operations.

That they are always on that focal conic may, however, be readily established by an easy application of the preceding principles, and without the necessity of having recourse to so painful a process. Let us first consider for a moment the case of curves.

For the two *focal* fixed points, let two circles be substituted of arbitrary radii and in any positions; the question will then come, to find the locus of a point in a plane whose tangential distances from the two fixed circles are always equal respectively to the distances of a variable point on the fixed right line from the two directive points, that is the sum or difference, as the case may be, of whose tangential distances from the circles is constant:* this locus also, there is no difficulty in seeing, will be always of the second order, and therefore the essential features of the question will not in the present case be altered by the substitution of the circles for the fixed points.

Now the locus, whatever be its nature, it is easy to see *à priori*, will have always double contact with each of the circles; for, every point which it may have in common with either circle being such that its tangential distance from that circle is nothing, its tangential distance from

* To this may be immediately reduced another and very recently discovered method of generating curves of the second order, viz. by the motion of a point in a plane, such that, tangents being drawn from it to two fixed, similar, similarly placed, and equal ellipses taken arbitrarily in that plane, and the points of contact being joined with the centres, the sum or difference of the two quadrilateral areas thus formed by the pairs of tangents and the pairs of radii shall be constant. For if, as is obviously possible, we project orthographically upon another plane parallel to the common direction of their minor axes the two ellipses into two circles of equal radii, then obviously will the sum or difference of the projected areas be also constant, that is, the sum or difference of the tangents to the circles drawn from the projection of the moveable point, multiplied by the common length of the equal radii, or the sum or difference of the tangents themselves, will be constant; hence the projected locus will be always a conic having double contact with each of the two circles, and therefore the original locus will be also a conic having double contact with each of the fixed ellipses.

the other circle must, in order to fulfil the conditions of the question, be equal to the constant distance between the two fixed directive points: hence, on either circle there can be but two points in common with the locus: for the curve locus of points whose tangential distance from the other circle is equal to that or to any other constant quantity, being obviously a third circle concentric with the latter, can intersect the former in never more than two points, real or imaginary; and the chord joining these points, always real whether the points themselves are real or not, will obviously be always perpendicular to the line joining the centres of the two circles.

Again, no point on the locus, whatever be its nature, can ever lie within either of the circles, for the tangential distance of every *internal* point from that circle would be imaginary, while the distances of *every* point on the fixed right line from the two directive fixed points are both always real.

Hence the locus, having but two points in common with each circle, and never passing within either of them, will have always double contact with both, real or imaginary; and therefore, being of the second order, when these circles both dwindle into points, those points will be the foci of the resulting curve. And here we may observe, that were we ignorant of that well-known fact, and acquainted with the curves of the second order only through the equations which represent them, the method just given would be perhaps the simplest by which we could proceed to establish the property.

Returning now to the case of surfaces, let the three *focal* fixed points be replaced by three spheres of arbitrary radii and fixed in any positions, then the question will become, to find the locus of a point in space whose tangential distances from these three spheres are always equal respectively to the three distances of a variable point in the plane of the directive triangle from the three fixed directive points: this locus also like the former, there will be no additional difficulty in shewing, will be always of the second order,*

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* For, denoting by p, q, r the radii of the three spheres, we have obviously but to substitute in the equation (2) (of note page 150), for l^2, m^2, n^2, the quantities $l^2 - p^2$, $m^2 - q^2$, $n^2 - r^2$ respectively, and that equation will obviously remain unaltered as to its order in x, y, z.

The same equation shews also immediately that in Jacobi's more general method, in which the fixed plane of the variable point is not taken as coinciding with the particular plane of the directive triangle, but is considered as being any fixed plane whatever, the resulting locus will be

and therefore in this case also the substitution of the spheres for the fixed points will not alter the essential features of the question.

Here also, as in the case of curves, it is easy to see *à priori* that the locus, whatever be its nature, will always have double contact, real or imaginary, with each of the three spheres; and moreover, that being of the second order, the peculiar and distinguishing nature of its double contact with all three will always be such, that the six circles of intersection which in pairs accompany its double contact with each sphere will always, whatever be the nature of the contact itself, be all imaginary; for, every point which that surface may have in common with any one of the three spheres being necessarily such that its tangential distance from that sphere is nothing, its tangential distances from the other two spheres must, in order to fulfil the conditions of the question, be equal respectively to the corresponding fixed distances of the directive point corresponding to that sphere from the other two directive points; therefore it can have never more than two points in common with each sphere, and even those may as often be both imaginary as real: for the loci of the extremities of tangents of any constant lengths to any two of the spheres being obviously always a pair of concentric spheres, these loci will therefore intersect the third sphere in a pair of circles, of which the points of intersection will obviously be the only points on that sphere which can satisfy the necessary conditions, and of which the planes will intersect in a right line passing always through those two points, real or imaginary, and being always perpendicular to the plane passing through the centres of the three spheres, which last is also evident from the obvious symmetry of the locus, whatever be its nature, on both sides of that plane.

Again, no point of the locus can ever pass within any one of the three spheres, for then the tangential distance of every *internal* point from that sphere would be imaginary, while the three distances of *every* point in the plane of the

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always a surface of the second order; for, projecting orthographically the directive triangle upon that plane, whatever be its position, and denoting by p, q, r the lengths of the projecting perpendiculars from the three angular points, we have, obviously, but to substitute in the equation for l^2, m^2, n^2 the quantities l^2+p^2, m^2+q^2, n^2+r^2 respectively, l, m, n being now the distances of the variable point from the three fixed angular points of the projected triangle, and, as before, its order in x, y, z will manifestly remain unchanged.

directive triangle from the three angular points of that triangle are always all real.

The locus therefore lying always altogether outside, and having never more than two points in common, has always double contact, real or imaginary, with each of the three spheres; and therefore, being always of the second order, when these spheres all dwindle into points, those points will be always on a focal conic of the resulting surface.

Hence, in M. Jacobi's method of generating surfaces of the second order, the three fixed *focal* points are always all on the focal conic of the resulting surface in the principal plane of that surface in which they all three lie;* and moreover, the peculiar nature of the double contact being always such that the two intersecting circles in the general state of each touching sphere are always both imaginary, the particular focal conic on which they all lie will be always a *modular,* and therefore never an *umbilicar* focal.

The property of curves of the second order converse to the familiar method of generating those curves by the motion of a point whose distances from two fixed points are always equal respectively to the distances of a variable point in a right line from two fixed points in the same, is (as is well known) an immediate consequence of the modular method of generating curves of that order: for, taking the two foci, and the two corresponding directrices of any curve of the second order considered as generated by the modular method, and drawing through any point taken arbitrarily on the curve a line parallel to either of the two directing right lines, the two distances of that point from the two foci will then have always respectively the same constant ratio to the two corresponding intercepts of the parallel line, included between the point and the two corresponding directrices, but the two directrices being always parallel to each other, the portion of the parallel line intercepted between them will be always of invariable length, whatever be the position of the arbitrarily assumed point on the curves: hence, if on a fixed right line two fixed points be taken distant from each other by an interval equal to that intercepted portion, diminished or increased as the case may be in the modular ratio of the curve, then may that curve be considered as also generated by the motion of a point whose distances from

* Hence, the names "foci" and "focal triangle", which for the sake of convenience and distinction we have ventured to employ, are not altogether inapplicable.

its two foci are always equal respectively to the two distances of a variable point in the fixed right line, from its two fixed points; and hence the latter method of generating curves of the second order is an immediate consequence from, and is implicitly contained in, the modular method.

Similarly, the property of surfaces of the second order converse to M. Jacobi's method of generating those surfaces is also an immediate consequence of the modular method: for, taking arbitrarily *any three* fixed points on a modular focal conic of any surface of the second order considered as generated by the modular method, with the three fixed directrices corresponding to these three foci, and drawing through any point taken arbitrarily on the surface a plane parallel to either of the two directive planes, the three distances of that point from the three fixed foci will then have always respectively the same constant ratio to the three corresponding distances of the same point from the three points in which the parallel plane intersects the three corresponding directrices; but the three directrices being fixed and always parallel to each other, the triangle formed by connecting the three points in which they are met by the parallel plane will obviously be always invariable both in magnitude and figure, whatever be the position of the arbitrarily assumed point on the surface. Hence, if in a fixed plane three fixed points be taken forming a triangle similar and equal to the triangle intercepted between the three directrices by a plane parallel to either directive plane with its sides all diminished or increased, as the case may be, in the modular ratio of the surface, then, since the three distances of every pair of corresponding points in any two similar triangles from their corresponding angles respectively are always to each other in the same direct ratio as the corresponding sides of the triangles themselves, the surface may obviously be considered as also generated by the motion of a point whose distances from the three fixed foci (which may be *any three* modular foci whatever on the same focal conic) are always equal respectively to the three corresponding distances of a variable point in the fixed plane of the increased or diminished triangle from the three fixed angular points of that triangle. And hence M. Jacobi's method of generating surfaces of the second order is an immediate consequence from, and is implicitely contained in, the modular method.

We have already seen that in M. Jacobi's method of generation the fixed *focal* triangle is always inscribed in the

focal conic of the resulting surface in the principal plane of that surface passing through its three angular points; similarly, we now see from the above, that a triangle similar to the second fixed triangle, which we have ventured to call the *dirigent* or *directive* triangle, is always inscribed in the corresponding dirigent cylinder of the resulting surface, in a plane parallel to either directive plane of that surface, and having its three similar angles placed respectively on the three directrices of the corresponding angular points of the focal triangle, and that the common ratio of the sides of that inscribed triangle to the corresponding sides of the dirigent or directive triangle is always equal to the inverse of the corresponding modular ratio of the generated surface.*

From the obvious property that the plane rectilinear figure polar with respect to a circle of any triangle inscribed in or circumscribed about a concentric circle is always a similar triangle circumscribed about or inscribed in a third concentric circle, it is easy to see that if in M. Jacobi's method of generation the two fixed triangles be similar to each other, and have their similar the corresponding pairs of angular points, then will the resulting surface of the second order be always of revolution round the perpendicular to the plane of the focal triangle which passes through the centre of the circle circumscribed to that triangle; hence, being always a *modular* surface of revolution, it must be either an oblate spheroid or a circular hyperboloid of one sheet, concentric with the circle circumscribing the focal triangle, and having that circle for its focal circle, the spheroid obviously corresponding always to the case of the focal being less than the directive triangle, and the hyperboloid always to the opposite case.

In the particular case when the two triangles are not only similar but also equal to each other, then is it obvious, *à priori*, that the resulting surface will be infinitely flat, and either portion indifferently of the plane of the focal triangle bounded by the circle circumscribing that triangle: but this is exactly what ought then to take place, that particular surface being the well-known limit, and transition state between a modular ellipsoid and hyperboloid of revolution when the magnitude of the axis of revolution changing from real to imaginary passes through zero.

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* Hence the name "dirigent" or "directive," as applied to that triangle, is not altogether inapplicable.

If either of the two given triangles, the focal or the dirigent, be evanescent, that is, if its three angular points be given infinitely near to, or rather coinciding with each other, then shall we get the two limits to the same two surfaces, when in the one the focal circle dwindles into nothing, and when in the other the focal remaining of finite value the dirigent circle vanishes, viz. in the case of the ellipsoid, a sphere with the evanescent focal triangle as centre, and in the case of the hyperboloid, an infinitely slender circular cylinder perpendicular to the plane of the finite focal triangle, and passing through the centre of the circle circumscribing that triangle. These two cases are also evident *à priori*, for if in any case the focal triangle dwindle into a point, then will *every* point of the locus be equally distant from the three coinciding angular points of that triangle, and therefore so must the variable point in the plane of the directive triangle from the three angles of that triangle; hence the resulting surface must be a sphere with its centre at the focal triangle, and with its radius equal to that of the circle which circumscribes the directive triangle; and if in any case the directive triangle dwindle into a point, then must *every* variable point in its plane be equally distant from its three coincident angular points, and therefore so must also every point on the locus from the three angles of the finite focal triangle; hence, in that case the locus must be a right line perpendicular to the plane of the focal triangle, and passing through the centre of the circle which circumscribes that triangle.

In all other cases, when the two given triangles are either dissimilar, or when even being similar their similar are not given as their corresponding pairs of angular points; to be able to state anything accurately respecting the nature and species of the resulting surface, we must have recourse to the general equation in note page 150, which of course holds in every possible case and includes every possible variety. In general there is no difficulty in shewing, that, if m be the modular ratio corresponding to the principal plane of the focal triangle in the generated surface, and ϕ the angle which its directive planes make with that principal plane, then will the given ratio of the area of the focal to that of the directive triangle be always expressed by the product $m \,.\, m \sec \phi$: but this merely telling us that if the focal have a greater area than the directive triangle, the resulting surface cannot be either an ellipsoid or an elliptic paraboloid, and if it have a lesser, that the surface can not be either an hyperbolic paraboloid

or an hyperboloid of one sheet with its focal *ellipse* in the plane of the focal triangle, is of little use towards determining the nature or species of the surface in general: and of this the reason is obvious, for with the same two given triangles, and with therefore the same ratio of areas, we could for different combinations of the three pairs of corresponding points have no less than six different resulting surfaces, and these obviously would not in all cases be all of the same nature or species.

Trinity College, Dublin, Feb. 26, 1847.

Note.—Since the above paper was written, the author has learned that in his substitution of the sphere for the fixed point in the umbilicar method of generation, he has been anticipated by Professor Chasles; but it appears that illustrious mathematician did not seem to have considered the principle as of any particular value. It was on the occasion of the umbilicar method having been independently discovered and developed by M. Amiot, but not until after Mr. Salmon had published the first principles of that method in his Examination-papers in October 1842, and subsequently in the Dublin University Calendar for 1843, that Professor Chasles gave the same extension contained in the above paper: and it would seem, from his not having included in his extension the modular method, that he could not at that period have been aware that the long sought-for method of generating the surfaces of the second order, which should be equally simple in its principles, and equally general in its results with the well-known and familiar method of generating the curves of that order, had been actually discovered and developed by Professor MacCullagh; although long previous to that period its author had published an abstract of his theory in the *Proceedings of the Royal Irish Academy*, for the year 1837, vol. I. p. 89.

THEOREMS ON THE LINES OF CURVATURE OF AN ELLIPSOID.

By MICHAEL ROBERTS, Fellow of Trinity College, Dublin.

THE following theorems furnish a remarkable extension of an elementary property of conic sections to the lines of curvature of an ellipsoid.

1. If from the interior umbilics of a line of curvature two geodesic lines are drawn to the same point on the curve, the product of the tangents of the halves of the angles which they make with the arc of the principal ellipse joining the umbilics is constant.

2. If from one interior and one exterior umbilic, two geodesic lines are drawn to the same point of a line of

curvature, the ratio of the tangents of the halves of the angles which they make with the arc of the principal ellipse joining the umbilics is constant.

The demonstration of these properties rests on a theorem due to M. Gauss, which, as it is not generally known, it will be well to enunciate explicitly.

"Assuming, for the rectification of curves traced on any surface, the formula

$$ds^2 = d\rho^2 + P^2 d\omega^2,$$

where ρ is the geodesic radius vector of the element ds, and ω the angle which this radius vector makes with a fixed geodesic line passing through the pole; then P will be in general a function of two independent variables ρ, ω, such that

$$\frac{d^2P}{d\rho^2} + \frac{P}{RR'} = 0 \dots\dots\dots\dots\dots (1),$$

where R, R' are the radii of curvature of the surface corresponding to the element ds."

Let us now suppose that a geodesic line drawn from the umbilic O meets a given line of curvature in the point T: at T let us draw a tangent to the geodesic arc $OT(\rho)$ until it meets the plane of the umbilics in the point S. This point, as M. Chasles has demonstrated, belongs to the focal hyperbola of the surface; and, consequently, the normal drawn to the ellipsoid at T pierces the plane of the umbilics in a point F which is situated on the tangent drawn to the focal hyperbola at S. If then we put $TS = t$, $TF = n$, and the angle $TSF = \alpha$, we have evidently

$$t \tan \alpha = n.$$

But if p is the perpendicular let fall from the centre of the ellipsoid on the tangent plane at T, and b the mean semiaxis of the surface, then

$$np = b^2,$$

so that

$$t \tan \alpha = \frac{b^2}{p} \dots\dots\dots\dots\dots (2).$$

Now let T' be the point infinitely near to T on the geodesic line OT, and let us suppose that the tangent to this line at T' pierces the focal hyperbola in a point S' consecutive to S; we have then

$$\{T'S' - \text{geodesic arc } OT'\} - \{TS - \text{geodesic arc } OT\} = d(t - \rho),$$

whence, if $\delta\epsilon$ is the indefinitely small angle between the lines $T'S'$, TS,

$$d(t-\rho)=\frac{t\delta\epsilon}{\tan\alpha},$$

or, from equation (2),

$$d(t-\rho)=\frac{b^2\delta\epsilon}{p\tan^2\alpha}.$$

Now if γ be the radius of curvature of the geodesic line corresponding to the element $d\rho$, and D the semidiameter of the ellipsoid parallel to this element, then

$$d\rho=\gamma\delta\epsilon,\quad \gamma p=D^2,$$

so that

$$d(t-\rho)=\frac{b^2d\rho}{D^2\tan^2\alpha};$$

or, from equation (2),

$$d(t-\rho)=\frac{p^2t^2d\rho}{b^2D^2}.$$

Now if a and c are respectively the greatest and least semi-axes of the surface, the theorem of M. Joachimsthal gives

$$pD=ac,$$

whence

$$d(t-\rho)=\frac{p^4t^2d\rho}{a^2b^2c^2}.$$

But if R, R' are the radii of curvature of the ellipsoid at T, we have, by the theorem of M. Ch. Dupin,

$$RR'=\frac{a^2b^2c^2}{p^4},$$

from which we get

$$d(t-\rho)=\frac{t^2\,d\rho}{RR'},$$

or

$$\frac{dt}{d\rho}=1+\frac{t^2}{RR'}\quad\ldots\ldots\ldots\ldots\ldots\ (3).$$

Now if y is the distance of T from the plane of the umbilics

$$\frac{y}{t}=\frac{dy}{d\rho},$$

and, by substituting the value of t derived from this equation in equation (3), this latter becomes

$$\frac{d^2y}{d\rho^2}+\frac{y}{RR'}=0.$$

Comparing this equation with equation (1), it is evident that for a geodesic arc passing through the umbilic of an ellipsoid

$$P = y\phi(\omega),$$

and as in the neighbourhood of the umbilic the surface is spherical, there is no difficulty in seeing that

$$\phi(\omega) = \frac{1}{\sin \omega},$$

where ω is the angle made by the geodesic vector OT with the elliptic arc joining the umbilics. Whence we derive

$$P = \frac{y}{\sin \omega}.$$

Now if i be the angle under which the geodesic radius vector OT cuts one of the lines of curvature passing through T, we have

$$Pd\omega = \tan i\delta\rho,$$

$\delta\rho$ being the difference between two consecutive radii vectores of the line of curvature: and if we denote by the same letters marked with accents the corresponding quantities for the other umbilic,

$$P'd\omega' = \tan i'\delta\rho'.$$

But I have already shewn in a memoir inserted in the *Journal de Mathématiques*, that $i = i'$: and also that, according as the umbilics are both interior, or one exterior and one interior, we have

$$\delta\rho + \delta\rho' = 0, \quad \delta\rho - \delta\rho' = 0,$$

whence, in the same circumstances,

$$Pd\omega \pm P'd\omega' = 0,$$

or

$$\frac{d\omega}{\sin \omega} \pm \frac{d\omega'}{\sin \omega'} = 0,$$

a formula which demonstrates the theorems which I have announced in the commencement of this paper.

The function P, which appears to be of importance in the theory of curves traced on the surface of the ellipsoid, possesses several curious properties; among which I may mention the following.

"If a right cylinder be circumscribed to an ellipsoid, the value of P for all geodesic arcs terminated by the curve of contact and the remote umbilic is constant and equal to the mean semiaxis of the surface."

In a memoir which will shortly appear in the *Journal de Mathématiques,* I have deduced this property, as well as those already proved, by expressing the constant introduced by M. Jacobi in the second integral of the differential equation of the geodesic line as a function of the angle ω. I may also observe that in the analytical expression for P we light upon the remarkable functions Θ, H, which are of such importance in the theory of the elliptic integrals.

ON THE VALUE OF $\left(\frac{d}{dx}\right)^\theta x^0$ WHEN θ IS A POSITIVE PROPER FRACTION.

By the Rev. W. Center.*

Having in a former paper attempted to shew that the admission of the differential operation

$$\left(\frac{d}{dx}\right)^\theta \frac{1}{x^m} = (-1)^\theta \frac{\Gamma(m+\theta)}{\Gamma m} \frac{1}{x^{m+\theta}}$$

within the limits $m > 0$ and $(m+\theta) > 0$, necessarily involves the complete theory of general differentiation; it has since occurred to me that the question at issue may be ultimately reduced to the following form—what is $\left(\frac{d}{dx}\right)^\theta x^0$ when θ is a positive proper fraction? The subject in this form admits of being treated generally, and affords a clear view of the present state of the question.

Let $\phi(n+\theta, m)$ express the general differential coefficient of $\left(\frac{d}{dx}\right)^{n+\theta} x^m$, where θ is a positive proper fraction, and m and n positive integers. Then

$$\left(\frac{d}{dx}\right)^\theta x^0 = \phi(\theta, 0)\, x^{-\theta}.$$

Prop. 1. According as $\phi(\theta, 0)$ is nothing or finite, so also is $\left(\frac{d}{dx}\right)^{n+\theta} x^0$.

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* [This article is, by desire of the author, printed before the other paper alluded to at the commencement, which will appear in the next Number of the *Journal*.—Ed.]

For, since $\left(\frac{d}{dx}\right)^\theta x^0 = \phi(\theta, 0)\, x^{-\theta}$,

then $\left(\frac{d}{dx}\right)^{n+\theta} x^0 = \phi(\theta, 0)\left(\frac{d}{dx}\right)^n x^{-\theta}$

$$= \phi(\theta, 0)(-\theta)(-1-\theta)\ldots\ldots(-n+1-\theta)\, x^{-n-\theta},$$

or $\left(\frac{d}{dx}\right)^{n+\theta} x^0 = \phi(\theta, 0)(-1)^n\, \theta(1+\theta)\ldots\ldots(n-1+\theta)\, x^{-n-\theta}$,

whence the truth of the proposition is manifest.

PROP. 2. According as $\phi(\theta, 0)$ is nothing or finite, so also is $\left(\frac{d}{dx}\right)^\theta x^m$.

Since $\left(\frac{d}{dx}\right)^\theta x^0 = \phi(\theta, 0)\, x^{-\theta}$,

then $\left(\frac{d}{dx}\right)^{-m+\theta} x^0 = \phi(\theta, 0)\left(\frac{d}{dx}\right)^{-m} x^{-\theta}$

$$= \frac{\phi(\theta, 0)\, x^{m-\theta}}{(1-\theta)(2-\theta)\ldots(m-\theta)}.$$

But $\left(\frac{d}{dx}\right)^{-m+\theta} x^0 = \left(\frac{d}{dx}\right)^\theta \left(\frac{d}{dx}\right)^{-m} x^0 = \left(\frac{d}{dx}\right)^\theta \frac{x^m}{\Gamma(m+1)}$;

hence $\left(\frac{d}{dx}\right)^\theta x^m = \frac{\phi(\theta, 0)\,\Gamma(m+1)\, x^{m-\theta}}{(1-\theta)(2-\theta)\ldots(m-\theta)}$.

Now θ being a proper fraction, the denominator is finite; hence $\left(\frac{d}{dx}\right)^\theta x^m$ is nothing or finite, according as is $\phi(\theta, 0)$.

PROP. 3. If $\phi(\theta, 0)$ be finite, then $\left(\frac{d}{dx}\right)^\theta x^{-m}$ is infinite, m being a positive integer; but if $\phi(\theta, 0)$ be nothing, then $\left(\frac{d}{dx}\right)^\theta x^{-m}$ is either finite or nothing.

By Prop. 1, when $\phi(\theta, 0)$ is finite, so also is $\left(\frac{d}{dx}\right)^{m+\theta} x^0$.

But $\left(\frac{d}{dx}\right)^m x^0 = 0.(-1)(-2)\ldots(-m+1)\, x^{-m} = 0.(-1)^{m-1}\Gamma(m).x^{-m}$,

and $\left(\frac{d}{dx}\right)^{m+\theta} x^0 = 0.(-1)^{m-1}\Gamma(m)\left(\frac{d}{dx}\right)^\theta x^{-m}$.

Now in order that the second member may be finite, it is plainly necessary that $\left(\frac{d}{dx}\right)^{\theta} x^{-m}$ be infinite; but if, again, $\phi(\theta, 0)$ be nothing, the second member has to become nothing; a condition which is satisfied by $\left(\frac{d}{dx}\right)^{\theta} x^{-m}$ being either finite or nothing.

PROP. 4. According as $\phi(\theta, 0)$ or (which amounts to the same thing) $\phi(1-\theta, 0)$ is nothing or finite, so $\left(\frac{d}{dx}\right)^{\theta} x^{\theta}$ is infinite or finite.

Since $\left(\frac{d}{dx}\right)^{1-\theta} x^0 = \phi(1-\theta, 0)\, x^{\theta-1}$; perform the operation $\left(\frac{d}{dx}\right)^{-1+\theta}$ on both sides, and we have

$$x^0 = \frac{\phi(1-\theta, 0)}{\theta} \left(\frac{d}{dx}\right)^{\theta} x^{\theta}, \text{ and } \left(\frac{d}{dx}\right)^{\theta} x^{\theta} = \frac{\theta}{\phi(1-\theta, 0)};$$

so that the truth of the proposition is manifest.

The properties established in these propositions point to the possible existence of *two* distinct systems of fractional differentiation, according to the nature of the fundamental assumption as to the value of $\left(\frac{d}{dx}\right)^{\theta} x^0$ when θ is a positive proper fraction.

Two such systems have been given; the one by M. Liouville, of the form before given, and the other by Dr. Peacock, of the form $\left(\frac{d}{dx}\right)^{\theta} x^m = \frac{\Gamma(1+m)}{\Gamma(1+m-\theta)} x^{m-\theta}$.

The characteristic of the former is its implication of the principle $\left(\frac{d}{dx}\right)^{\theta} x^0 = 0$; from which it follows that

$$\left(\frac{d}{dx}\right)^{1+\theta} x^0 = 0 \ldots \left(\frac{d}{dx}\right)^{n+\theta} x^0 = 0; \left(\frac{d}{dx}\right)^{\theta} x^{\theta} = \infty;$$

as also that $\left(\frac{d}{dx}\right)^{\theta} x^m = 0$, when m is integer, and $\left(\frac{d}{dx}\right)^{\theta} x^{-m} =$ finite.

The latter system is characterised by the implied admission that $\left(\frac{d}{dx}\right)^{\theta} x^0$ is finite; (for here we have $\left(\frac{d}{dx}\right)^{\theta} x^0 = \frac{x^{-\theta}}{\Gamma(1-\theta)}$,

which is finite so long as θ is a positive proper fraction); from which it also follows that

$$\left(\frac{d}{dx}\right)^{1+\theta} x^0, \ldots \left(\frac{d}{dx}\right)^{n+\theta} x^0, \quad \left(\frac{d}{dx}\right)^{\theta} x^m, \ldots \left(\frac{d}{dx}\right)^{n+\theta} x^m \text{ and } \left(\frac{d}{dx}\right)^{\theta} x^{\theta}$$

are all finite, while $\left(\frac{d}{dx}\right)^{\theta} x^{-m}$ is infinite.

As both these formulæ comport themselves in exact accordance with the *general* properties before established, it would be in vain to search for points of discordance *within* either of the respective systems, for the purpose of establishing a ground of preference. On the one side, it has been customary to assume $\left(\frac{d}{dx}\right)^{\theta+1} x^0 = 0$ as an axiom in the new science; but, from the foregoing considerations, it is sufficiently plain that this assumption is a virtual adoption of the system of M. Liouville in preference to the other. Again, on the other side, the adoption of $\left(\frac{d}{dx}\right)^{\theta} x^{\theta}$ as the fundamental form for *all* positive values of θ, necessarily involves the admission that $\left(\frac{d}{dx}\right)^{\theta} x^{\theta}$ is *finite* when θ is fractional, a postulate shewn *à priori*, in Prop. 4, to rest on the prior assumption that $\phi(\theta, 0)$ is finite.

The whole question is therefore now plainly reduced to this, what is $\left(\frac{d}{dx}\right)^{\theta} x^0$ when θ is a positive proper fraction? for when this point is settled, we shall have determined at the same time which of the two systems we *must* adopt. And as the subject has been shewn *à priori* to branch out into the two distinct systems from the point $\left(\frac{d}{dx}\right)^{\theta} x^0$, our investigations must commence at this very point. Let us then for a moment suppose θ a positive integer; then, by successive differentiation,

$$\left(\frac{d}{dx}\right)^{\theta} x^0 = 0.(-1)(-2)\ldots(-\theta+1)\,x^{-\theta},$$

or

$$\left(\frac{d}{dx}\right)^{\theta} x^0 = 0.(-1)^{\theta-1}\,\Gamma(\theta).x^{-\theta}.$$

Now this is the solution when θ is integer; and we cannot (with our present knowledge) conceive the *form* to be dif-

ferent when θ is a positive proper fraction. The conclusion then appears inevitable, that for *all* positive values of θ, $\left(\frac{d}{dx}\right)^\theta x^0 = 0$.

Let us adopt this conclusion as the fundamental principle of our system, and proceed by means of it to establish the form of $\left(\frac{d}{dx}\right)^\theta \varepsilon^{ax}$, for *all* positive values of θ.

Since $$\varepsilon^{ax} = 1 + ax + \frac{a^2x^2}{1.2} \ldots + \frac{a^nx^n}{1.2\ldots n} + \&c.$$

and $$\frac{a^nx^n}{1.2\ldots n} = a^n \left(\frac{d}{dx}\right)^{-n} x^0, \text{ we have}$$

$$\varepsilon^{ax} = x^0 + a\left(\frac{d}{dx}\right)^{-1} x^0 + a^2\left(\frac{d}{dx}\right)^{-2} x^0 + a^3\left(\frac{d}{dx}\right)^{-3} x^0 + \&c.$$

$$= \left\{1 + a\left(\frac{d}{dx}\right)^{-1} + a^2\left(\frac{d}{dx}\right)^{-2} + a^3\left(\frac{d}{dx}\right)^{-3} + \&c.\right\} x^0,$$

or $$\varepsilon^{ax} = \left(\frac{d}{dx} - a\right)^{-1} \frac{d}{dx} x^0. \text{ Operate with } \left(\frac{d}{dx}\right)^\theta,$$

and $\left(\frac{d}{dx}\right)^\theta \varepsilon^{ax} = \left(\frac{d}{dx} - a\right)^{-1} \left(\frac{d}{dx}\right)^{\theta+1} x^0$; and since $\left(\frac{d}{dx}\right)^{\theta+1} x^0 = 0$,

we have $$\left(\frac{d}{dx}\right)^\theta \varepsilon^{ax} = \left(\frac{d}{dx} - a\right)^{-1} 0;$$

the solution of which gives

$$\left(\frac{d}{dx}\right)^\theta \varepsilon^{ax} = C.\, \varepsilon^{ax}.$$

The constant C is independent of x, and must be some function of a and θ; and in order to determine its form, we require only to know that form in some particular case of the problem, such as when θ is a positive integer, and then $C = a^\theta$. Hence we have generally for all positive values of θ,

$$\left(\frac{d}{dx}\right)^\theta \varepsilon^{ax} = a^\theta . \varepsilon^{ax} \ldots\ldots\ldots\ldots (a).$$

With the aid of this general form and the well-known auxiliary equation $\frac{\Gamma(m)}{x^m} = \int_0^\infty \varepsilon^{-ax} a^{m-1}\, da$, we can now pass on without any discontinuity to the establishment of the first

case of M. Liouville's formula $\left(\frac{d}{dx}\right)^\theta \frac{1}{x^m} = (-1)^\theta \frac{\Gamma(m+\theta)}{\Gamma(m)} \frac{1}{x^{m+\theta}}$ comprehended within the limits $m > 0$ and $m + \theta > 0$, and thence to the complete theory of general differentiation.

But it would still be of considerable importance, could we establish the said case of the formula without the aid of the general property (a); for we should then have two *independent* formulæ of operation, which would be mutually illustrative. The following considerations will enable us to effect this.

By Prop. 3, we know that $\left(\frac{d}{dx}\right)^\mu \frac{1}{x}$ *may* be finite (certainly not infinite) for all positive values of μ, when $\left(\frac{d}{dx}\right)^\mu x^0 = 0$.

But $$\left(\frac{d}{dx}\right)^\mu \frac{1}{x} = \phi(\mu, -1) \frac{1}{x^{\mu+1}},$$

and $$\left(\frac{d}{dx}\right)^{\mu+\theta} \frac{1}{x} = \phi(\mu, -1) \left(\frac{d}{dx}\right)^\theta \frac{1}{x^{\mu+1}};$$

but $$\left(\frac{d}{dx}\right)^{\mu+\theta} \frac{1}{x} = \phi(\mu + \theta, -1) \frac{1}{x^{\mu+\theta+1}},$$

hence $$\left(\frac{d}{dx}\right)^\theta \frac{1}{x^{\mu+1}} = \frac{\phi(\mu+\theta, -1)}{\phi(\mu, -1)} \frac{1}{x^{\mu+\theta+1}}.$$

The question now occurs, what is $\phi(\mu, -1)$? When μ is integer, it is $(-1)(-2)\ldots\ldots(-\mu) = (-1)^\mu \surd(\mu+1)$; and in the present state of our knowledge we cannot conceive the form to be different when μ is fractional.

Hence $$\left(\frac{d}{dx}\right)^\theta \frac{1}{x^{\mu+1}} = (-1)^\theta \frac{\Gamma(\mu+\theta+1)}{\Gamma(\mu+1)} \frac{1}{x^{+\theta+1}};$$

in which we may without scruple write generally m for $\mu + 1$, whence

$$\left(\frac{d}{dx}\right)^\theta \frac{1}{x^m} = (-1)^\theta \frac{\Gamma(m+\theta)}{\Gamma(m)} \frac{1}{x^{m+\theta}} \ldots\ldots\ldots (b),$$

subject to the proper limits of the function $m > 0$, and $m + \theta > 0$.

Observe that the formulæ (a) and (b) are now independent, and also both of the same class as connected with the same fundamental principle $\left(\frac{d}{dx}\right)^\theta x^0 = 0$; we are then at liberty to exhibit their mutual consistency by instances of verification.

The following is one. By common integration we have

$$\int_0^\infty \varepsilon^{-ax^2}\, x\, dx = \frac{1}{2a},$$

with the limit $a > 0$. Perform the operation indicated by $\left(\frac{d}{da}\right)^\theta$, on the first side by (a) and on the second by (b); then, omitting the factor $(-1)^\theta$ common to both sides, we have

$$\int_0^\infty \varepsilon^{-ax^2}\, x^{2\theta+1}\, dx = \frac{\frac{1}{2}\Gamma(\theta+1)}{a^{\theta+1}};$$

put $\theta = -\frac{1}{2}$, $$\int_0^\infty \varepsilon^{-ax^2}\, dx = \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{a^{\frac{1}{2}}},$$

put $a = 1$, $$\int_0^\infty \varepsilon^{-x^2}\, dx = \tfrac{1}{2}\pi^{\frac{1}{2}}.$$

These integrals are well known.

Longside, Mintlaw, April 1, 1848.

NOTE ON A RESULT OF ELIMINATION.

By the Rev. George Salmon.

The object of the present note being to present a result of elimination in a form adapted to some geometrical applications, I premise a few elementary remarks to explain the nature of these applications to readers unfamiliar with the subject.

Let $$at^m + mbt^{m-1} + \frac{m.m-1}{1.2}ct^{m-2} + \&c. = T = 0$$

be the equation of a tangent to a curve (where a, b, c, are linear functions of the coordinates, and t a variable parameter); then this curve is evidently one of the m^{th} class, or such that m tangents can be drawn to it from any point; since, if we substitute the coordinates of any point in the given equation, we obtain an equation of the m^{th} degree to determine t.

Since every point on the curve is the intersection of two consecutive tangents, every point on the curve must satisfy the condition that $T = 0$, considered as a function of t, should have two equal roots; the equation of the curve $U = 0$ is

therefore found by eliminating t between $T=0$, and $\frac{dT}{dt}=0$.*

Since the result of this elimination is of the degree $2(m-1)$ in a, b, c, &c., we perceive that the curve is not the most general curve of the m^{th} class; these curves being of the degree $m(m-1)$.

The same thing can be shewed by proving that this curve must necessarily have multiple points. A double point, for example, is a point at which two distinct pairs of consecutive tangents intersect. If therefore we join the two conditions that the equation T should have two pairs of equal roots, these conditions are the equations of two curves whose intersection determines a number of points $\{= 2(m-2)(m-3)\}$, which will be double points on the curve U.

Again, a cusp is a point at which three consecutive tangents intersect. If therefore we form the two conditions that the equation T should have three equal roots, we shall have the equation of two curves whose intersection will determine $3(m-2)$ points which will be cusps on the curve U.

Thus we see that the class of this curve of the $2m-2$ degree having $3(m-2)$ cusps and $2(m-2)(m-3)$ double points

$$= \{(2m-2)(2m-3) - 9(m-2) - 4(m-2)(m-3)\} = m.\dagger$$

Let us now proceed to space of three dimensions, and suppose $T=0$ to be the equation of a tangent plane to a developable surface, whose equation is $U=0$. The equations $T=0$, $\frac{dT}{dt}=0$, determine any of the generating lines of the surface, which is evidently such that m tangent planes can be drawn to it from any point. By making $z=0$, in a, b, c,

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* It might be supposed that the double tangents of the curve ought to enter as factors into the result of elimination, since from every point of these, two coincident tangents can be drawn to the curve. However, it is not true that for these points the equation to determine t will have equal roots, but that the *same* right line will be represented by two *different* values of t.

† The same results might be obtained by shewing that the curve obtained by eliminating t between

$$at^m + mbt^{m-1} + \frac{m.m-1}{1.2}c.t^{m-2} + \&c. = 0,$$

$$a't^m + mb't^{m-1} + \frac{m.m-1}{1.2}c't^{m-2} + \&c. = 0,$$

is of the $2m^{\text{th}}$ degree having the maximum number, $(m-1)(2m-1)$ of double points; and that if $a'=b$, $b'=c$, &c., then, the total number of double points remaining the same, the maximum number of these $\{=3(m-1)\}$ will become cusps.

&c., any section is found from the preceding to be a curve of the $2m-2$ degree, having $3(m-2)$ cusps, and $2(m-2)(m-3)$ double points.

The same two equations which in the last case determined the cusps are now the equations of a curve of the $3(m-2)$ order which is the *cuspidal edge* of the surface or the curve whose tangents generate the surface. And we see that the surface will have also a *nodal line* of the $2(m-2)(m-3)$ degree, such that the points in which any plane cuts it will be double points in the corresponding section.

A simple illustration of such nodal lines occurs in the case of the developable generated by the tangents to the curve of intersection of two surfaces of the second degree, for in this case the nodal line breaks up into four plane curves. Take any of the four planes whose poles with regard to both surfaces are the same, and it is easy to see that from the point where any tangent to the curve meets this plane, there can also be drawn a second tangent to the curve.

In precisely the same manner we find the multiple *points* of the developable surface. The three conditions that the equation T should have four equal roots, determine the cuspidal points $\{=4(m-3)\}$ of the edge of regression; besides which the surface has other multiple points where the edge of regression is intersected by the nodal line.

I come now to the immediate object of this note, namely, the particular case where $m=4$. Mr. Cayley has (in Crelle's Journal) presented the result of elimination between

$$at^4+4bt^3+6ct^2+4dt+e=0,$$

and its differential, in such a form as to exhibit the cuspidal edge of the surface: viz.

$$(ae-4bd+3c^2)^3=27(ace+2bcd-ad^2-eb^2-c^3)^2.$$

Now the same equation can also be presented in such a form as to exhibit the nodal line of the surface, viz.

$$(ae^2-9ec^2+2bde+6cd^2)(a^2e-9ac^2+2abd+6b^2c)$$
$$-16(2bd^2-3bce+ade)(2b^2d-3acd+abe)-27(ad^2-eb^2)^2=0.$$

In order to deduce from the equation in this shape the existence of the nodal line, it is necessary to shew that the five members into which it is divided, represent surfaces having a common line of intersection.*

* [Representing the equation by $PQ-16UV-27W^2=0$, this is in fact saying that P, Q, W are each of them of the form $\lambda U+\mu V$, (where however λ, μ are not integral functions), and this being the case, the

Now the second pair of these can be put into the form

$$\frac{3ac - 2b^2}{a} = \frac{3ce - 2d^2}{e} = \frac{ad}{b} = \frac{be}{d} = \frac{ae + 2bd}{3c};$$

whence can be deduced the other three equations, and also the two $\quad 2b^3 + a^2d - 3abc = 0, \quad 2d^3 + e^2b - 3cde = 0.$

Now when from the identity of the three equations

$$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'},$$

we infer that the three curves

$$AB' = BA', \qquad AC' = CA', \qquad BC' = B'C,$$

have common points of intersection, it does not follow that *all* the points of intersection of these curves are common, for any point whose coordinates make one of those fractions assume the form $\frac{0}{0}$ will satisfy two of these equations, but not the third; and, if A, B, C; A', B', C' be right lines, the three curves just mentioned will have only three points common. So in like manner the seven surfaces we are considering although they have a common line of intersection, yet have not all their points of intersection common. For example, the surfaces $ad^2 - eb^2$, $2bd^2 - 3bec + ade$ have common, the line ab, the line bd which being a double line on one of them counts for two in the order of their intersection, and the line ed which also counts for two, since along it the surfaces are touched by the plane e: but these lines are not common to all the other surfaces, there remains therefore but a curve of the fourth degree, common to all the surfaces.

It is natural to enquire whether this curve of the fourth degree can be represented as the intersection of two surfaces of the second degree. Now any point on the curve may be conveniently represented by the equations,

$$b = \mu d, \quad a = \mu^2 e, \quad d = \rho e,$$

where ρ is determined by an equation of the form

$$\rho^2 - 2(A\mu + B)\rho + C\mu^2 + D\mu + E = 0.$$

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equation reduces itself to the form $AU^2 + 2BUV + CV^2 + 0$, (A, B, C being also not integral), which shews that $U = 0$, $V = 0$, or at least that this system divested of an extraneous curve represents the nodal line. The actual values are

$$P = \frac{1}{b}(3cU + eV), \quad Q = \frac{1}{d}(aU + 3cV), \quad W = \frac{1}{3c}(bU - dV) \;].$$

If these values be substituted in the general equation of the second degree, and ρ eliminated, we obtain an equation of the 8th degree in μ, the nine terms of which put separately $= 0$ should enable us to determine surfaces of the second degree containing the given curve. I am however only able to write down the equation of one such surface, viz.

$$C^2a^2 + Cb^2 + Ed^2 + E^2e^2 + 2(AD - BC)ad$$
$$+ 2(BD - AE)be - Dbd - 2BEde - 2ACab + 2(CE - D^2)ae = 0.$$

Trinity College, Dublin, April 25, 1848.

ON GEOMETRICAL RECIPROCITY.

By Arthur Cayley.

The fundamental theorem of reciprocity in plane geometry may be thus stated.

"The points and lines of a plane P may be considered as corresponding to the lines and points of a plane P' in such a manner that to a set of points in a line in the first figure, there corresponds a set of lines through a point in the second figure, (namely through the point corresponding to the line); and to a set of lines through a point in the first figure, there corresponds a set of points in a line in the second figure, namely in the line corresponding to the point."

And from this theorem, without its being in any respect necessary further to particularize the nature of the correspondence, or to consider in any manner the relative position of the two planes, an endless variety of propositions and theories may be deduced, as, for instance, the quality of all theorems which relate to the purely descriptive properties of figures, the theory of the singular points and tangents of curves, &c.

Suppose, however, that the two planes coincide, so that a point may be considered indifferently as belonging to the first or to the second figure: an entirely independent series of propositions (which, properly speaking, form no part of the general theory of reciprocity) result from this particularization. In general, the line in the second figure which corresponds to a point considered as belonging to the first figure, and the line in the first figure which corresponds to the same point considered as belonging to the second figure, will not be iden-

tical; neither will the point in the second figure which corresponds with a point considered as belonging to the first figure, and the point in the first figure which corresponds to the same line considered as belonging to the second figure, be identical.

In the particular case where these lines and points are respectively identical (the identity of the lines implies that of the points and *vice versâ*) we have the theory of "reciprocal polars." Here, where it is unnecessary to define whether the points or lines belong to the first or second figures, the line corresponding to a point and the point corresponding to a line are spoken of as the polar of the point and the pole of the line, or as reciprocal polars.

"The points which lie in their respective polars are situated in a conic, to which the polars are tangents." Or stating the *same* theorem conversely,

"The lines which pass through their respective poles are tangents to a conic, the points of contact being the poles."

To determine the polar of a point, let two tangents be drawn to the conic through this point, the points of contact are the poles of the tangents; hence the line joining them is the polar of the point of intersection of the tangents, *i.e.* "The polar of a point is the line joining the points of contacts of the tangents which pass through the point."

Conversely, and by the same reasoning,

"The pole of a line is the intersection of the tangents at the points where the line meets the conic."

The actual geometrical constructions in the several cases where the point is within or without the conic, or the line does or does not intersect the conic, do not enter into the plan of the present memoir.

Passing to the general case where the lines and points in question are not identical, which I should propose to term the theory of "Skew Polars" (Polaires Gauches), we have the theorem,

"Considering the points in the first figure which are situated in their respective corresponding lines in the second figure, or the points in the second figure which are situated in their respective corresponding lines in the first figure, in either case the points are situated in the same conic (which will be spoken of as the 'pole conic'), and the lines are tangents to the same conic (which will be spoken of as the 'polar conic'), and these two conics have a double contact." A theorem which is evidently identical with the converse theorem.

The corresponding lines to a point in the pole conic are the tangents through this point to the polar conic; viz. one of these tangents is the corresponding line when the point is considered as belonging to the first figure, and the other tangent is the corresponding line when the point is considered as belonging to the second figure.

The corresponding points to a tangent of the polar conic are the points where this line intersects the pole conic; viz. one of these points is the corresponding point when the line is considered as belonging to the first figure, and the other is the corresponding point when the line is considered as belonging to the second figure.

Let i be a point in the pole conic, and when i is considered as belonging to the first figure, let iI_1 be considered as the corresponding line in the second figure (I_1 being the point of contact on the polar conic).

Then if j be another point in the pole conic, in order to determine which of the tangents is the line in the second figure which corresponds to j considered as a point of the first figure, let iI_2 be the other tangent through I: the points of contact of the tangents through j may be marked with the letters J_1, J_2, in such order that I_1J_2, I_2J_1 meet in the line of contact of the two conics, and then jJ_1 is the required corresponding line. Again, I and i, as before, if B be a tangent to the polar conic, then, marking the point of contact as J_1, let J_2 be so determined that I_1J_2, I_2J_1 meet in the line of contact of the conics, the tangent to the polar conic at J_2 will meet the pole conic in one of the points where it is met by the line B, and calling this point j, B considered as belonging to the second figure will have j for its corresponding point in the first figure. Similarly, if the point of contact had been marked J_2, J_1 would be determined by an analogous construction, and the tangent at J_1 would meet the pole conic in one of the points where it is met by the line B (viz. the other point of intersection); and representing this by j', B considered as belonging to the first figure would have j' for its corresponding point in the second figure, *i.e.* considered as belonging to the second figure it would have j for its corresponding point in the first figure (the same as before).

Similar considerations apply in the case where a tangent A of the polar conic, considered as belonging to one of the figures, has for its corresponding point in the other figure one of its points of intersection with the pole conic; in fact, if A represent the line iI_1, then A, considered as belonging

to the second figure has i for its corresponding, point in the first figure, which shews that this question is identical with the former one.

[To appreciate these constructions it is necessary to bear in mind the following system of theorems, the third and fourth of which are the polar reciprocals of the first and second.

"If there be two conics having a double contact, such that K is the line joining the points of contact, and k the point of intersection of the tangents at the points of contact."

1. If two tangents to one of the conics meet the other in i, i_1 and j, j_1 respectively, then, properly selecting the points j, j_1, the lines ij_1, i_1j meet in K. And

2. The line joining the points of intersection of the tangents at i, j_1, and of the tangents at i_1, j passes through k. Also

3. If from two points of one of the conics, tangents be drawn touching the other in the points I, I_1 and J, J_1, then, properly selecting the points J, J_1, the lines IJ_1, I_1J meet in K. And

4. The line joining the points of intersection of the tangents at I_1J_1 and of the tangents at I_1, J passes through k.

These theorems are in fact particular cases of two theorems relating to two conics, having a double contact with a given conic.

It may be remarked also that the corresponding points to a tangent of the pole conic are the points of contact of the tangents to the polar conic which pass through the point of contact of the given tangent, and the corresponding lines to a point of the polar conic are the tangents to the pole conic at the points where it is intersected by the tangent at the point in question.

We have now to determine the corresponding lines to a given point and the corresponding points to a given line, which is immediately effected by means of the preceding results.

Thus, if the point be given,

"Through the point draw tangents to the polar conic, meeting the pole conic in A_1, A_2 and B_1, B_2, (so that A_1B_2 and A_2B_1, intersect on the line joining the points of contact of the conics), then A_2B_2 and A_1B_1 are the required lines."

In fact A_1, B_1 and A_2, B_2 are pairs of points corresponding to the two tangents, so that A_1B_1 and A_2B_2 are the lines which correspond to their point of intersection, *i.e.* to the given point, and similarly for the remaining constructions. Again,

"Through the point draw tangents to the pole conic, and from the points of contact draw tangents to the polar conic, touching it in a_1, a_2 and β_1, β_2 (so that $a_1\beta_2$ and $a_2\beta_1$ intersect on the line joining the points of contact of the conics), then $a_1\beta_1$ and $a_2\beta_2$ are the required lines."

So that A_1, B_1, a_1, β_1 are situated in the same line, and also A_2, B_2, a_2, β_2.

Again, if the line be given,

"Through the points where the line meets the pole conic draw tangents to the polar conic C_1, C_2 and D_1, D_2, (so that the points C_1D_2 and C_2D_1 lie on a line passing through the intersection of the tangents at the points of contact of the tangents), then C_1D_1 and C_2D_2 are the required points".

Again,

"At the points where the line meets the polar conic draw tangents meeting the pole conic, and let γ_1, γ_2 and δ_1, δ_2 be the tangents to the pole conic at these points, (so that the points $\gamma_1\delta_2$ and $\gamma_2\delta_1$ lie on a line through the intersection of the tangents at the points of contact of the conics), then γ_1, δ_1 and γ_2, δ_2 are the required points"; so that C_1, D_1, γ_1, δ_1 pass through the same point and also C_2, D_2, γ_2, δ_2.

"The preceding constructions have been almost entirely taken from Plücker's "System der Analytischen Geometrie", §3, Allgemeine Betrachtongen über Coordmauen Bestimmung. I subjoin analytical demonstrations of some of the theorems in question.

Using x, y, z to determine the position of a variable point, and putting for shortness

$$\xi = ax + a'y + a''z,$$
$$\eta = bx + b'y + b''z,$$
$$\zeta = cx + c'y + c''z.$$

Then if the position of a point be determined by the coordinates a, β, γ, the equation of one of the corresponding lines is

$$a\xi + \beta\eta + \gamma\zeta = 0,$$

(that of the other is obtainable from this by writing a, b, c; a', b', c'; a'', b'', c'', for a, a', a''; b, b', b''; c, c', c''). Hence if the point lies in the corresponding line, this equation must be satisfied by putting a, β, γ for x, y, z; or, substituting x, y, z in the place of a, β, γ, the point must lie in the conic

$$U = ax^2 + b'y^2 + c''z^2 + (b'' + c')\,yz + (c + a'')\,zx + (a' + b)\,xy = 0,$$

(which equation is evidently not altered by interchanging

the coefficients, as above). Again, determining the curve traced out by the line $\alpha\xi + \beta\eta + \gamma\zeta = 0$, when α, β, γ are connected by the equation into which $U = 0$ is transformed by the substitution of these letters for x, y, z; these results

$$V = -\begin{vmatrix} & \xi, & \eta, & \zeta \\ \xi, & 2a, & a' + b, & a'' + c \\ \eta, & a' + b, & 2b', & b'' + c' \\ \zeta, & a'' + c, & b'' + c', & 2c'' \end{vmatrix} = 0,$$

which is also a conic. It only remains to be seen that the conics $U = 0$, $V = 0$ have a double contact. Writing for shortness

$$\nabla = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

it may be seen by expansion that the following equation is identically true,

$$V = 4\nabla U - [x\,(ab'' - a''b + a'c - ac') + y\,(b'c - bc' + b''a' - b'a'') + z\,(c''a' - c'a'' + cb'' - c''b)]^2,$$

which proves the property in question.

Suppose the equations of the two conics to be given, and let it be required to determine the corresponding lines to the point defined by the coordinates α, β, γ.

Writing, to abbreviate,

$$\left\{\begin{aligned} U &= Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy, \\ U_0 &= A\alpha^2 + B\beta^2 + C\gamma^2 + 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta, \\ W &= A\alpha x + B\beta y + C\gamma z + H(\beta z + \gamma y) + G(\gamma x + \alpha z) + H(\alpha y + \beta x), \\ P &= lx + my + nz, \\ P_0 &= l\alpha + m\beta + n\gamma, \\ K &= ABC - AF^2 - BG^2 - CH^2 + 2FGH. \end{aligned}\right.$$

$$\mathfrak{A} = BC - F^2,\ \mathfrak{B} = CA - G^2,\ \mathfrak{C} = AB - H^2,$$

$$\mathfrak{F} = GH - AF,\ \mathfrak{G} = HF - BG,\ \mathfrak{H} = FG - CH.$$

$$\Theta = \mathfrak{A}l^2 + \mathfrak{B}m^2 + \mathfrak{C}n^2 + 2\mathfrak{F}mn + 2\mathfrak{G}nl + 2\mathfrak{H}lm.$$

$$\begin{aligned} \square = {} & (\mathfrak{A}l + \mathfrak{H}m + \mathfrak{G}n)\,(\gamma y - \beta z), \\ & + (\mathfrak{H}l + \mathfrak{B}m + \mathfrak{F}n)\,(\alpha z - \gamma x), \\ & + (\mathfrak{G}l + \mathfrak{F}m + \mathfrak{C}n)\,(\beta x - \alpha y). \end{aligned}$$

Suppose $U = 0$ represents the equation of the polar conic, $U - P^2 = 0$ that of the pole conic. The two tangents drawn to the polar conic are represented by $UU_0 - W^2 = 0$, and by

determining k in such a way that $UU_0 - W^2 - k(U - P^2)$ may divide into factors the equation

$$UU_0 - W^2 - k(U - P^2) = 0,$$

represents the lines passing through the points of intersection of the tangents with the pole conic. Thus if $k = U_0$, the equation reduces itself to $U_0P^2 - W^2 = 0$, or $W = \pm\sqrt{(U_0)}P$, the equation of two straight lines each of which passes through the point of intersection of the lines $P = 0$, $W = 0$, (*i.e.* of the line of contact of the conics, and the ordinary polar of the point with respect to the polar conic); these are in fact the lines A_1B_2, A_2B_1 intersecting in the line of contact. The remaining value of k is not easily determined, but by a somewhat tedious process I have found it to be

$$K(U_0 - P_0^2) : (\Theta - K).$$

In fact, substituting the value, it may be shewn that

$$\square^2 + K(PP_0 - W)^2 = K(U_0 - P_0^2)(U - P^2) + (\Theta - K)(UU_0 - W^2).$$

which is an equation of this required form. To verify this, we have, by a simple reduction,

$$\Theta(UU_0 - W^2) - \square^2 = K(UP_0^2 - 2WPP_0 + U_0P^2).$$

Or, writing for shortness $\gamma y - \beta z = \xi$, $\alpha z - \gamma x + \eta$, $\beta x - \alpha y = \zeta$,

$$\begin{aligned}
&(\mathfrak{A}l^2 + \mathfrak{B}m^2 + \mathfrak{C}n^2 + 2\mathfrak{F}mn + 2\mathfrak{G}nl + \mathfrak{H}lm) \\
&\qquad (\mathfrak{A}\xi^2 + \mathfrak{B}\eta^2 + \mathfrak{C}\zeta^2 + 2\mathfrak{F}\eta\zeta + 2\mathfrak{G}\zeta\xi + 2\mathfrak{H}\xi\eta) \\
&- [\mathfrak{A}l\xi + \mathfrak{B}m\eta + \mathfrak{C}n\zeta + \mathfrak{F}(n\eta + m\zeta) + \mathfrak{G}(l\zeta + n\xi) + \mathfrak{H}(m\xi + l\eta)]^2 \\
&= K\{A(m\zeta - n\eta)^2 + B(n\xi - l\zeta)^2 + C(l\eta - m\xi)^2 \\
&\qquad + 2F(n\xi - l\zeta)(l\eta - m\xi) + 2G(l\eta - m\xi)(m\zeta - n\eta) \\
&\qquad\qquad + 2H(m\zeta - n\eta)(n\xi - l\zeta)\},
\end{aligned}$$

which is easily seen to be identically true.

ON CERTAIN ALGEBRAIC FUNCTIONS.

[*Being a Supplement to the paper at pp.* 267–273 *of Vol. II.*]

By James Cockle.

X. In my former paper on this subject I omitted to notice a modification of which our processes are susceptible when the number that indicates the degree of a homogeneous

function is capable of decomposition into factors. This omission I now propose to supply.

Let us first consider the expression

$$f^4(u_m),$$

where 4 (the degree of the function) $= 2 \times 2$. In this case, instead of reducing the given expression by the processes which I have employed in paragraph III,* we may, starting with the new equation of finite differences,

$$u_{x+1} = 3.2^{2u_x} \ldots\ldots\ldots\ldots\ldots (12),$$

reduce it to the form of a sum of m squares.

Assume, as we are at liberty to do, that

$$f^4(u_m) = \Xi^4 + 2A\Xi^3 + (A^2 + 2B)\,\Xi^2 + C\Xi + D;$$

then the right-hand side of this equation may be put under the form

$$(\Xi^2 + A\Xi + B)^2 + (C - 2AB)\,\Xi + D - B^2,$$

hence, adopting the notation which I used in the last volume, and making

$$\Xi^2 + A\Xi + B = \mathfrak{P}_1,$$

we have $f^4(u_m) = \mathfrak{P}_1^2 + f^3(u_m - 1)\,\Xi + f^4(u_m - 1) \ldots (13);$

but since, by (12), $u_m - 1 = 3.2^{2u_{m-1}} - 1,$

we see that $f^3(u_m - 1)$ may, by the processes of paragraph II,† be made to vanish without involving us in elevation of degree, and that the right-hand side of (13) may be reduced to the form

$$\mathfrak{P}_1^2 + f^4(u_{m-1});$$

so, by successive deductions, we may arrive at

$$f^4(u_m) = \mathfrak{P}_1^2 + \mathfrak{P}_2^2 + . + \mathfrak{P}_x^2 + f^4(u_{m-x}):$$

now, as in paragraph III, let

$$u_{m-m} = u_0 = 2;$$

then $f^4(u_m)$ may be reduced to the form of a sum of m squares, and we shall not be encumbered with superfluous quantities.

Let $$r = n \times p \times q \times . \times .,$$

then the transformation of $f^r(u_m)$ to the forms

$$\Sigma(\mathfrak{P}^n), \quad \Sigma(\mathfrak{P}^p), \quad \Sigma(\mathfrak{P}^q), . , . ,$$

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* *Cambridge and Dublin Mathematical Journal*, Vol. II. p. 269.

† Ibid, pp. 267—268.

admit of special discussions analogous to the foregoing: but it is not worth while to enter into them here.

The reader will be pleased to erase the "1 +" from the fourth line of p. 273 of the last volume of this work. The two upper lines of the expression for Υ should be

$$3.2^{2u}\begin{bmatrix}4, & 5, .., & r-1, & r\\ \delta, & \epsilon, .., & b, & a\end{bmatrix}$$

2, Church-Yard Court, Temple,
14th Feb. 1848.

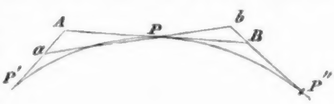
NOTE ON THE MINIMUM VALUE OF THE AREA OF A POLYGON CIRCUMSCRIBED ABOUT A GIVEN REENTERING CURVE.

By the Rev. HARVEY GOODWIN.

IN page 142, Vol. II., of the *Cambridge Mathematical Journal* an algebraical proof is given of the theorem, that the area of the polygon circumscribing a given reentering curve is least when the sides of the polygon are bisected in the points of contact.

The following demonstration of this elegant theorem seems worthy of notice for its simplicity.

The condition of maximum for a function of any number of variables is this, that if each of the variables independently receive a very small change, there shall be no change in the value of the function.



Let now the figure represent a portion of a curve figure circumscribed by a polygon, and let the side AB of the polygon be bisected in P. Suppose the side AB to be very slightly changed in position, so as to assume the position ab, which touches the curve at a point very near to P, and may be supposed ultimately to pass through P. Then we have ultimately

$$\frac{\text{triangle } APa}{\text{triangle } BPb} = \frac{AP^2}{PB^2} = 1,$$

since AB is bisected in P.

Hence the area of the figure $P'ABP''$ is the same as that of $P'abP''$; that is, the area of the polygon is not changed by

slightly varying the position of the points of contact. And therefore (since maximum is evidently out of the question) the area is a minimum.

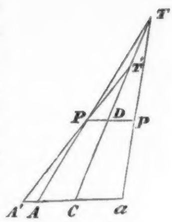
It is easy to deduce this theorem; the area of a polygon circumscribing an ellipse is least when each side is parallel to the chord of contact of the adjacent sides; of which the theorem, that those parallelograms are least which have their sides parallel to conjugate diameters is a particular case. I mention this theorem in order to observe, that the same mode of investigation which has been used above is applicable to shew that the areas of all such circumscribing polygons is the same, as is well known in the case of the parallelogram. For, if the points of contact be slightly shifted, it will appear almost exactly as above, that as much is added to the area on one side of each point of contact as is taken away on the other, and therefore that the area is not changed in magnitude; and this being the case however many times we suppose the points shifted, the absolute constancy of the area is concluded.

The same method of investigation would shew that if a curve surface be encased in an envelope composed of plane figures having an even number of sides, then the volume of the enveloping surface is least when the plane figures composing it are regular figures, and the points of contact are the centres of those regular figures. For in this case any line drawn through the point of contact will bisect the face of the envelope, and therefore the volume will not be altered by supposing the face to have a small twist about this bisecting line. For example, a cube will be the smallest figure of six faces which can envelope a sphere. If the faces have an odd number of sides the method does not appear to be applicable.

Theorems may be enunciated concerning the ellipsoid analogous to those concerning the polygons enveloping an ellipse, which have been given above. Thus, the volume of the tetrahedron, enveloping an ellipsoid, will be least when each face is parallel to the plane of contact of the other three.

Suppose we have a cone enveloping an ellipsoid, then the cone of minimum volume will have its base parallel to the plane of contact, but this will not entirely determine the cone; we may however apply a method similar to that already used to shew that the plane of contact will divide the length of the cone in two parts in the ratio of $1:2$. For let the plane of contact be perpendicular to the plane of the paper, and let Pp be the line of intersection and let TAa be the section of the enveloping cone, also let $TP = y$, $PA = x$.

Suppose the generating lines of the cone to be turned through a small angle about the points of contact, and let $A'PT'$ be the new position of the line APT. Then a certain portion will be cut off from the upper part of the cone and a certain portion added to the lower, and in the minimum state of the cone these must be equal; it is not difficult to see, from Guldin's properties of the centre of gravity, that this condition of equality will be satisfied if



$$x^2 \times \left(y + \frac{2x}{3}\right) = y^2 \times \frac{y}{3},$$

or
$$\frac{y^3}{x^3} - 3\frac{y}{x} = 2,$$

or
$$\frac{y}{x} = 2,$$

which is the condition above enunciated.

When the preceding note was written, I was not aware that the method had even been published. I find however that in his recent work on Conic Sections Mr. Salmon has applied it to shew that the tangent to the interior of two similar, similarly placed, and concentric conics cuts off a constant area from the exterior conic.

THE CALCULUS OF LOGIC.

By George Boole.

In a work lately published,* I have exhibited the application of a new and peculiar form of Mathematics to the expression of the operations of the mind in reasoning. In the present essay I design to offer such an account of a portion of this treatise as may furnish a correct view of the nature of the system developed. I shall endeavour to state distinctly those positions in which its characteristic distinctions consist, and shall offer a more particular illustration of some features which are less prominently displayed in the

* *The Mathematical Analysis of Logic, being an Essay towards a Calculus of Deductive Reasoning.* Cambridge, Macmillan; London, G. Bell.

original work. The part of the system to which I shall confine my observations is that which treats of categorical propositions, and the positions which, under this limitation, I design to illustrate, are the following:

(1) That the business of Logic is with the relations of classes, and with the modes in which the mind contemplates those relations.

(2) That antecedently to our recognition of the existence of propositions, there are laws to which the conception of a class is subject,—laws which are dependent upon the constitution of the intellect, and which determine the character and form of the reasoning process.

(3) That those laws are capable of mathematical expression, and that they thus constitute the basis of an interpretable calculus.

(4) That those laws are, furthermore, such, that all equations which are formed in subjection to them, even though expressed under functional signs, admit of perfect solution, so that every problem in logic can be solved by reference to a general theorem.

(5) That the forms under which propositions are actually exhibited, in accordance with the principles of this calculus, are analogous with those of a philosophical language.

(6) That although the symbols of the calculus do not depend for their interpretation upon the idea of quantity, they nevertheless, in their particular application to syllogism, conduct us to the quantitative conditions of inference.

It is specially of the two last of these positions that I here desire to offer illustration, they having been but partially exemplified in the work referred to. Other points will, however, be made the subjects of incidental discussion. It will be necessary to premise the following notation.

The universe of conceivable objects is represented by 1 or unity. This I assume as the primary and subject conception. All subordinate conceptions of class are understood to be formed from it by limitation, according to the following scheme.

Suppose that we have the conception of any group of objects consisting of Xs Ys, and others, and that x, which we shall call an elective symbol, represents the mental operation of selecting from that group all the Xs which it contains, or of fixing the attention upon the Xs to the exclusion of

all which are not Xs, y the mental operation of selecting the Ys, and so on; then, 1 or the universe being the subject conception, we shall have

$x\,1$ or x = the class X,

$y\,1$ or y = the class Y,

$xy\,1$ or xy = the class each member of which is both X and Y,

and so on.

In like manner we shall have

$1 - x$ = the class not-X,

$1 - y$ = the class not-Y,

$x(1 - y)$ = the class whose members are Xs but not-Ys,

$(1 - x)(1 - y)$ the class whose members are neither Xs nor Ys,

&c.

Furthermore, from consideration of the nature of the mental operation involved, it will appear that the following laws are satisfied.

Representing by x, y, z, any elective symbols whatever,

$$x(y + z) = xy + xz \ldots\ldots\ldots\ldots (1),$$

$$xy = yx, \text{ \&c.} \ldots\ldots\ldots\ldots (2),$$

$$x^n = x, \text{ \&c.} \ldots\ldots\ldots\ldots (3).$$

From the first of these it is seen that elective symbols are distributive in their operation; from the second that they are *commutative*. The third I have termed the index law; it is peculiar to elective symbols.

The truth of these laws does not at all depend upon the nature, or the number, or the mutual relations, of the individuals included in the different classes. There may be but one individual in a class, or there may be a thousand. There may be individuals common to different classes, or the classes may be mutually exclusive. All elective symbols are distributive, and commutative, and all elective symbols satisfy the law expressed by (3).

These laws are in fact embodied in every spoken or written language. The equivalence of the expressions "good wise man" and "wise good man," is not a mere truism, but an assertion of the law of commutation exhibited in (2). And there are similar illustrations of the other laws.

With these laws there is connected a general axiom. We have seen that algebraic operations performed with elective

symbols represent mental processes. Thus the connexion of two symbols by the sign + represents the aggregation of two classes into a single class, the connexion of two symbols xy as in multiplication, represents the mental operation of selecting from a class Y those members which belong also to another class X, and so on. By such operations the conception of a class is modified. But beside this the mind has the power of perceiving relations of equality among classes. The axiom in question, then, is that *if a relation of equality is perceived between two classes, that relation remains unaffected when both subjects are equally modified by the operations above described.* (A). This axiom, and not "Aristotle's dictum," is the real foundation of all reasoning, the form and character of the process being, however, determined by the three laws already stated.

It is not only true that every elective symbol representing a class satisfies the index law (3), but it may be rigorously demonstrated that any combination of elective symbols $\phi(xyz..)$, which satisfies the law $\phi(xyz..)^n = \phi(xyz..)$, represents an intelligible conception,—a group or class defined by a greater or less number of properties and consisting of a greater or less number of parts.

The four categorical propositions upon which the doctrine of ordinary syllogism is founded, are

All Ys are Xs.	A,
No Ys are Xs.	E,
Some Ys are Xs.	I,
Some Ys are not Xs.	O.

We shall consider these with reference to the classes among which relation is expressed.

A. The expression All Ys represents the class Y and will therefore be expressed by y, the copula are by the sign =, the indefinite term, Xs, is equivalent to Some Xs. It is a convention of language, that the word Some is expressed in the subject, but not in the predicate of a proposition. The term Some Xs will be expressed by vx, in which v is an elective symbol appropriate to a class V, some members of which are Xs, but which is in other respects arbitrary. Thus the proposition A will be expressed by the equation

$$y = vx \qquad (4).$$

E. In the proposition, No Ys are Xs, the negative particle appears to be attached to the subject instead of to the

predicate to which it manifestly belongs.* We do not intend to say that those things which are not-Ys are Xs, but that things which are Ys are not-Xs. Now the class not-Xs is expressed by $1 - x$; hence the proposition No Ys are Xs, or rather All Ys are not-Xs, will be expressed by

$$y = v(1 - x) \ldots\ldots\ldots\ldots\ldots (5).$$

I. In the proposition Some Ys are Xs, or Some Ys are Some Xs, we might regard the Some in the subject and the Some in the predicate as having reference to the same arbitrary class V, and so write

$$vy = vx,$$

but it is less of an assumption to refrain from doing this. Thus we should write

$$vy = v'x \ldots\ldots\ldots\ldots\ldots (6),$$

v' referring to another arbitrary class V′.

O. Similarly, the proposition Some Ys are not-Xs, will be expressed by the equation

$$vy = v'(1 - x) \ldots\ldots\ldots\ldots\ldots (7).$$

It will be seen from the above that the forms under which the four categorical propositions A, E, I, O are exhibited in the notation of elective symbols are analogous with those of pure language, *i.e.* with the forms which human speech would assume, were its rules entirely constructed upon a scientific basis. In a vast majority of the propositions which can be conceived by the mind, the laws of expression have not been modified by usage, and the analogy becomes more apparent, *e.g.* the interpretation of the equation

$$z = x(1 - y) + y(1 - x),$$

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* There are but two ways in which the proposition, No Xs are Ys, can be understood. 1st, In the sense of All Xs are not-Y. 2nd, In the sense of It is not true that any Xs are Ys, *i.e.* the proposition "Some Xs are Ys" is false. The former of these is a single categorical proposition. The latter is *an assertion respecting a proposition*, and its expression belongs to a distinct part of the elective system. It appears to me that it is the latter sense, which is really adopted by those who refer the negative, *not*, to the copula. To refer it to the predicate is not a useless refinement, but a necessary step, in order to make the proposition truly a *relation between classes*. I believe it will be found that this step is really taken in the attempts to demonstrate the Aristotelian rules of distribution.

The transposition of the negative is a very common feature of language. Habit renders us almost insensible to it in our own language, but when in another language the same principle is differently exhibited, as in the Greek, οὔ φημι for φημὶ οὔ, it claims attention.

is, the class Z consists of all Xs which are not-Ys and of all Ys which are not-Xs.

General Theorems relating to Elective Functions.

We have now arrived at this step,—that we are in possession of a class of symbols x, y, z, &c. satisfying certain laws, and applicable to the rigorous expression of any categorical proposition whatever. It will be our next business to exhibit a few of the general theorems of the calculus which rests upon the basis of those laws, and these theorems we shall afterwards apply to the discussion of particular examples.

Of the general theorems I shall only exhibit two sets: those which relate to the development of functions, and those which relate to the solution of equations.

Theorems of Development.

(1) If x be any elective symbol, then

$$\phi(x) = \phi(1)\,x + \phi(0)\,(1 - x) \ldots\ldots\ldots (8),$$

the coefficients $\phi(1)$, $\phi(0)$, which are quantitative or common algebraic functions, are called the moduli, and x and $1 - x$ the constituents.

(2) For a function of two elective symbols we have

$$\phi(xy) = \phi(11)\,xy + \phi(10)\,x(1 - y) + \phi(01)(1 - x)\,y + \phi(00)(1 - x)(1 - y) \ldots (9),$$

in which $\phi(11)$, $\phi(10)$, &c. are quantitative, and are called the moduli, and xy, $x(1 - y)$, &c. the constituents.

(3) Functions of three symbols,

$$\begin{aligned}\phi(xyz) = {} & \phi(111)\,xyz + \phi(110)\,xy\,(1 - z) \\ & + \phi(101)\,x(1 - y)\,z + \phi(100)\,x(1 - y)(1 - z) \\ & + \phi(011)(1 - x)\,yz + \phi(010)(1 - x)\,y\,(1 - z) \\ & + \phi(001)(1 - x)(1 - y)\,z + \phi(000)(1 - x)(1 - y)(1 - z) \\ & \ldots\ldots\ldots\ldots (10),\end{aligned}$$

in which $\phi(111)$, $\phi(110)$, &c. are the moduli, and xyz, $xy(1 - z)$, &c. the constituents.

From these examples the general law of development is obvious. And I desire it to be noted that this law is a mere consequence of the primary laws which have been expressed in (1), (2), (3).

THEOREM. *If we have any equation $\phi(xyz\,..) = 0$, and fully expand the first member, then every constituent whose modulus does not vanish may be equated to* 0.

This enables us to interpret any equation by a general rule.

RULE. *Bring all the terms to the first side, expand this in terms of all the elective symbols involved in it, and equate to* 0 *every constituent whose modulus does not vanish.*

For the demonstration of these and many other results, I must refer to the original work. It must be noted that on p. 66, z has been, through mistake, substituted for w, and that the reference on p. 80 should be to Prop. 2.

As an example, let us take the equation

$$x + 2y - 3xy = 0 \;\ldots\ldots\ldots\ldots\; (11).$$

Here $\phi(xy) = x + 2y - 3xy$, whence the values of the moduli are $\phi(11) = 0, \quad \phi(10) = 1, \quad \phi(01) = 2, \quad \phi(00) = 0,$

so that the expansion (9) gives

$$x(1 - y) + 2y(1 - x) = 0,$$

which is in fact only another form of (11). We have, then, by the Rule

$$x(1 - y) = 0 \;\ldots\ldots\ldots\ldots\; (11),$$

$$y(1 - x) = 0 \;\ldots\ldots\ldots\ldots\; (12);$$

the former implies that there are no Xs which are not-Ys, the latter that there are no Ys which are not-Xs, these together expressing the full significance of the original equation.

We can, however, often recombine the constituents with a gain of simplicity. In the present instance, subtracting (12) from (11), we have

$$x - y = 0,$$

or

$$x = y,$$

that is, the class X is identical with the class Y. This proposition is equivalent to the two former ones.

All equations are thus of equal significance which give, on expansion, the same series of constituent equations, and *all are interpretable.*

General Solution of Elective Equations.

(1) The general solution of the equation $\phi(xy) = 0$, in which two elective symbols only are involved, y being the

one whose value is sought, is

$$y = \frac{\phi(10)}{\phi(10) - \phi(11)}\, x + \frac{\phi(00)}{\phi(00) - \phi(01)}\,(1 - x) \ldots (13).$$

The coefficients

$$\frac{\phi(10)}{\phi(10) - \phi(11)}, \quad \frac{\phi(00)}{\phi(00) - \phi(01)}$$

are here the moduli.

(2) The general solution of the equation $\phi(xyz) = 0$, z being the symbol whose value is to be determined, is

$$z = \frac{\phi(110)}{\phi(110) - \phi(111)}\, xy + \frac{\phi(100)}{\phi(100) - \phi(101)}\, x(1 - y)$$

$$+ \frac{\phi(010)}{\phi(010) - \phi(011)}\,(1 - x)\, y + \frac{\phi(000)}{\phi(000) - \phi(001)}\,(1 - x)(1 - y)$$

$$\ldots\ldots\ldots\ldots (14),$$

the coefficients of which we shall still term the moduli. The law of their formation will readily be seen, so that the general theorems which have been given for the solution of elective equations of two and three symbols, may be regarded as examples of a more general theorem applicable to all elective equations whatever. In applying these results it is to be observed, that if a modulus assume the form $\frac{0}{0}$ it is to be replaced by an arbitrary elective symbol w, and that if a modulus assume any numerical value except 0 or 1, the constituent of which it is a factor must be separately equated to 0. Although these conditions are deduced solely from the laws to which the symbols are obedient, and without any reference to interpretation, they nevertheless render the solution of every equation interpretable in logic. To such formulæ also *every question upon the relations of classes may be referred.* One or two very simple illustrations may suffice.

(1) Given $\quad yx = yz + x(1 - z) \ldots\ldots\ldots\ldots\ldots (a).$

The Ys which are Xs consist of the Ys which are Zs and the Xs which are not-Zs. Required the class Z.

Here $\quad \phi(xyz) = yx - yz - x(1 - z),$

$$\phi(111) = 0, \quad \phi(110) = 0, \quad \phi(101) = 0,$$

$$\phi(100) = -1, \quad \phi(011) = -1, \quad \phi(010) = 0,$$

$$\phi(001) = 0, \quad \phi(000) = 0\,;$$

and substituting in (14), we have

$$z = \tfrac{0}{0}xy + x(1-y) + \tfrac{0}{0}(1-x)(1-y)$$
$$= x(1-y) + wxy + w'(1-x)(1-y) \ldots\ldots (15).$$

Hence the class Z includes all Xs which are not-Ys, an indefinite number of Xs which are Ys, and an indefinite number of individuals which are neither Xs nor Ys. The classes w and w' being quite arbitrary, the indefinite remainder is equally so; it may vanish or not.*

Since $1-z$ represents a class, not-Z, and satisfies the index law

$$(1-z)^n = 1-z,$$

as is evident on trial, we may, if we choose, determine the value of this element just as we should determine that of z.

Let us take, in illustration of this principle, the equation $y = vx$, (All Ys are Xs), and seek the value of $1-x$, the class not-X.

Put $1-x=z$ then $y = v(1-z)$, and if we write this in the form $y - v(1-z) = 0$ and represent the first member by $\phi(vyz)$, v here taking the place of x, in (14), we shall have

$$\phi(111)=1,\quad \phi(110)=0,\quad \phi(101)=0,\quad \phi(100)=-1,$$
$$\phi(011)=1,\quad \phi(010)=1,\quad \phi(001)=0,\quad \phi(000)=0;$$

the solution will thus assume the form

$$z = \frac{0}{0-1}vy + \frac{-1}{-1-0}v(1-y) + \frac{1}{1-1}(1-v)y + \frac{0}{0-0}(1-v)(1-y),$$

or $1-x = v(1-y) + \tfrac{1}{0}(1-v)y + \tfrac{0}{0}(1-v)(1-y) \ldots\ldots (16).$

The infinite coefficient of the second term in the second member permits us to write

$$y(1-v) = 0 \ldots\ldots\ldots\ldots (17),$$

[* This conclusion may be illustrated and verified by considering an example such as the following.

Let x denote all steamers, or steam-vessels,

y . . . . armed vessels,

z . . . . vessels of the Mediterranean.

Equation (α) would then express that *armed steamers consist of the armed vessels of the Mediterranean and the steam-vessels not of the Mediterranean.* From this it follows—

(1) That there are no armed vessels except steamers in the Mediterranean.

(2) That all unarmed steamers are in the Mediterranean (since the steam-vessels not of the Mediterranean are armed). Hence we infer that *the vessels of the Mediterranean consist of all unarmed steamers; any number of armed steamers; and any number of unarmed vessels without steam.* This, expressed symbolically, is equation (15).]

the coefficient $\frac{0}{0}$ being then replaced by w, an arbitrary elective symbol, we have

$$1 - x = v(1 - y) + w(1 - v)(1 - y),$$

or $$1 - x = \{v + w(1 - v)\}(1 - y) \quad \ldots\ldots\ldots (18).$$

We may remark upon this result that the coefficient $v + w(1 - v)$ in the second member satisfies the condition

$$\{v + w(1 - v)\}^n = v + w(1 - v),$$

as is evident on squaring it. It therefore represents a *class*. We may replace it by an elective symbol u, we have then

$$1 - x = u(1 - y) \quad \ldots\ldots\ldots\ldots (19),$$

the interpretation of which is

All not-Xs are not-Ys.

This is a known transformation in logic, and is called conversion by contraposition, or negative conversion. But it is far from exhausting the solution we have obtained. Logicians have overlooked the fact, that when we convert the proposition All Ys are (some) Xs, into All not-Xs are (some) not-Ys there is a relation between the two (*somes*), understood in the predicates. The equation (18) shews that whatever may be that condition which limits the Xs in the original proposition,—the not-Ys in the converted proposition consist of all which are subject to the same condition, and of an arbitrary remainder which are not subject to that condition. The equation (17) further shews that there are no Ys which are not subject to that condition.

We can similarly reduce the equation $y = v(1 - x)$, No Ys are Xs, to the form $x = v'(1 - y)$ No Xs are Ys, with a like relation between v and v'. If we solve the equation $y = vx$ All Ys are Xs, with reference to v, we obtain the subsidiary relation $y(1 - x) = 0$ No Ys are not-Xs, and similarly from the equation $y = v(1 - x)$ (No Ys are Xs) we get $xy = 0$. These equations, which may also be obtained in other ways, I have employed in the original treatise. All equations whose interpretations are connected are similarly connected themselves, by solution or development.

On Syllogism.

The forms of categorical propositions already deduced are

$y = vx$,	All Ys are Xs,
$y = v(1 - x)$,	No Ys are Xs,
$vy = v'x$,	Some Ys are Xs,
$vy = v'(1 - x)$,	Some Ys are not-Xs,

whereof the two first give, by solution, $1 - x = v'(1 - y)$. All not-Xs are not-Ys, $x = v'(1 - y)$, No Xs are Ys. To the above scheme, which is that of Aristotle, we might annex the four categorical propositions

$1 - y = vx$, All not-Ys are Xs,

$1 - y = v(1 - x)$, All not-Ys are not-Xs,

$v(1 - y) = v'x$, Some not-Ys are Xs,

$v(1 - y) = v'(1 - x)$, Some not-Ys are not-Xs,

the two first of which are similarly convertible into

$1 - x = v'y$, All not-Xs are Ys,

$x = v'y$, All Xs are Ys,

or No not-Xs are Ys.

If now the two premises of any syllogism are expressed by equations of the above forms, the elimination of the common symbol y will lead us to an equation expressive of the conclusion.

Ex. 1. All Ys are Xs, $y = vx$,

All Zs are Ys, $z = v'y$,

the elimination of y gives

$$z = vv'x,$$

the interpretation of which is

All Zs are Xs,

the form of the coefficient vv' indicates that the predicate of the conclusion is limited by both the conditions which separately limit the predicates of the premises.

Ex. 2. All Ys are Xs, $y = vx$,

All Ys are Zs, $y = v'z$.

The elimination of y gives

$$v'z = vx,$$

which is interpretable into Some Zs are Xs. It is always necessary that one term of the conclusion should be interpretable by means of the equations of the premises. In the above case both are so.

Ex. 3. All Xs are Ys, $x = vy$,

No Zs are Ys, $z = v'(1 - y)$.

Instead of directly eliminating y let either equation be transformed by solution as in (19). The first gives

$$1 - y = u(1 - x),$$

u being equivalent to $v + w(1 - v)$, in which w is arbitrary. Eliminating $1 - y$ between this and the second equation of the system, we get

$$z = v'u(1 - x),$$

the interpretation of which is

No Zs are Xs.

Had we directly eliminated y, we should have had

$$vz = v'(v - x),$$

the reduced solution of which is

$$z = v'\{v + w(1 - v)\}(1 - x),$$

in which w is an arbitrary elective symbol. This exactly agrees with the former result.

These examples may suffice to illustrate the employment of the method in particular instances. But its applicability to the demonstration of general theorems is here, as in other cases, a more important feature. I subjoin the results of a recent investigation of the Laws of Syllogism. While those results are characterized by great simplicity and bear, indeed, little trace of their mathematical origin, it would, I conceive, have been very difficult to arrive at them by the examination and comparison of particular cases.

Laws of Syllogism deduced from the Elective Calculus.

We shall take into account all propositions which can be made out of the classes X, Y, Z, and referred to any of the forms embraced in the following system,

A,	All Xs are Zs.	A′,	All not-Xs are Zs.
E,	No Xs are Zs.	E′,	No not-Xs are Zs, or (All not-Xs are not-Zs).
I,	Some Xs are Zs.	I′,	Some not-Xs are Zs.
O,	Some Xs are not-Zs.	O′,	Some not-Xs are not-Zs.

It is necessary to recapitulate that quantity (universal and particular) and quality (affirmative and negative) are under-

stood to belong to the *terms* of propositions which is indeed the correct view.*

Thus, in the proposition All Xs are Ys, the subject All Xs is universal-affirmative, the predicate (some) Ys particular-affirmative.

In the proposition, Some Xs are Zs, both terms are particular-affirmative.

The proposition No Xs are Zs would in philosophical language be written in the form All Xs are not-Zs. The subject is universal-affirmative, the predicate particular-negative.

In the proposition Some Xs are not Zs, the subject is particular-affirmative, the predicate particular-negative. In the proposition All not-Xs are Ys the subject is universal-negative, the predicate particular-affirmative, and so on.

In a pair of premises there are four terms, viz. two subjects and two predicates; two of these terms, viz. those involving the Y or not-Y may be called the middle terms, the two others the extremes, one of these involving X or not-X, the other Z or not-Z.

The following are then the conditions and the rules of inference.

Case 1st. The middle terms of like quality.

Condition of Inference. One middle term universal.

Rule. Equate the extremes.

Case 2nd. The middle terms of opposite qualities.

1st. Condition of Inference. One extreme universal.

Rule. Change the quantity and quality of that extreme, and equate the result to the other extreme.

2nd. Condition of inference. Two universal middle terms.

Rule. Change the quantity and quality of either extreme, and equate the result to the other extreme.

I add a few examples,

1st. All Ys are Xs.
All Zs are Ys.

This belongs to Case 1. All Ys is the universal middle term. The extremes equated give All Zs are Xs, the stronger term becoming the subject.

2nd. $\left.\begin{array}{l}\text{All Xs are Ys}\\ \text{No Zs are Ys}\end{array}\right\} = \left\{\begin{array}{l}\text{All Xs are Ys.}\\ \text{All Zs are not-Ys.}\end{array}\right.$

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* When *propositions* are said to be affected with quantity and quality, the quality is really that of the *predicate*, which expresses the *nature* of the assertion, and the quantity that of the *subject*, which shews its extent.

This belongs to Case 2, and satisfies the first condition. The middle term is particular-affirmative in the first premise, particular-negative in the second. Taking All Zs as the universal extreme, we have, on changing its quantity and quality, Some not-Zs, and this equated to the other extreme gives

All Xs are (some) not-Zs = No Xs are Zs.

If we take All Xs as the universal extreme we get

No Zs are Xs.

3rd. All Xs are Ys.

Some Zs are not Ys.

This also belongs to Case 2, and satisfies the first condition. The universal extreme All Xs becomes, some not-Xs, whence

Some Zs are not-Xs.

4th. All Ys are Xs.

All not-Ys are Zs.

This belongs to Case 2, and satisfies the second condition. The extreme Some Xs becomes All not-Xs,

∴ All not-Xs are Zs.

The other extreme treated in the same way would give

All not-Zs are Xs,

which is an equivalent result.

If we confine ourselves to the Aristotelian premises A, E, I, O, the second condition of inference in Case 2 is not needed. The conclusion will not necessarily be confined to the Aristotelian system.

Ex. $\left.\begin{array}{l}\text{Some Ys are not Xs}\\ \text{No Zs are Ys}\end{array}\right\} = \left\{\begin{array}{l}\text{Some Ys are not-Xs.}\\ \text{All Zs are not-Ys.}\end{array}\right.$

This belongs to Case 2, and satisfies the first condition. The result is Some not-Zs are not-Xs.

These appear to me to be the ultimate laws of syllogistic inference. They apply to every case, and they completely abolish the distinction of figure, the necessity of conversion, the arbitrary and partial* rules of distribution, &c. If all logic

* Partial, because they have reference only to the quantity of the X, even when the proposition relates to the not-X. It would be possible to construct an exact counterpart to the Aristotelian rules of syllogism, by quantifying only the not-X. The system in the text is *symmetrical* because it is complete.

were reducible to the syllogism these might claim to be regarded as the rules of logic. But logic, considered as the science of the relations of classes has been shewn to be of far greater extent. Syllogistic inference, in the elective system, corresponds to elimination. But this is not the highest in the order of its processes. All questions of elimination may in that system be regarded as subsidiary to the more general problem of the solution of elective equations. To this problem all questions of logic and of reasoning, without exception, may be referred. For the fuller illustrations of this principle I must however refer to the original work. The theory of hypothetical propositions, the analysis of the positive and negative elements, into which all propositions are ultimately resolvable, and other similar topics are also there discussed.

Undoubtedly the final aim of speculative logic is to assign the conditions which render reasoning possible, and the laws which determine its character and expression. The general axiom (A) and the laws (1), (2), (3), appear to convey the most definite solution that can at present be given to this question. When we pass to the consideration of hypothetical propositions, the same laws and the same general axiom which ought perhaps also to be regarded as a law, continue to prevail; the only difference being that the subjects of thought are no longer classes of objects, but cases of the coexistent truth or falsehood of propositions. Those relations which logicians designate by the terms conditional, disjunctive, &c., are referred by Kant to distinct conditions of thought. But it is a very remarkable fact, that the expressions of such relations can be deduced the one from the other by mere analytical process. From the equation $y = vx$, which expresses the *conditional* proposition. "If the proposition Y is true the proposition X is true," we can deduce

$$yx + (1 - y)\,x + (1 - y)\,(1 - x) = 1,$$

which expresses the *disjunctive* proposition. "Either Y and X are together true, or X is true and Y false, or they are both false," and again the equation $y\,(1 - x) = 0$, which expresses a relation of coexistence, *viz.* that the truth of Y and the falsehood of X do not coexist. The distinction in the mental regard, which has the best title to be regarded as fundamental, is, I conceive, that of the affirmative and the negative. From this we deduce the direct and the inverse in operations, the true and the false in propositions, and the opposition of qualities in their terms.

The view which these inquiries present of the nature of language is a very interesting one. They exhibit it not as a mere collection of signs, but as a system of expression, the elements of which are subject to the laws of the thought which they represent. That those laws are as rigorously mathematical as are the laws which govern the purely quantitative conceptions of space and time, of number and magnitude, is a conclusion which I do not hesitate to submit to the exactest scrutiny.

ON A CERTAIN PERIODIC FUNCTION.

By Henry Wilbraham, B.A., Trinity College, Cambridge.

Fourier, in his Treatise on Heat, after discussing the equation

$$y = \cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \ldots\ldots \textit{ ad inf.},$$

says that, if x be taken for abscissa and y for ordinate, it will represent a locus composed of separate straight lines, of which each is equal to π, parallel to the axis of x, and at a distance $\frac{1}{4}\pi$ alternately above and below it, joined by perpendiculars which are themselves part of the locus of the equation. Some writers who have subsequently considered the equation, have stated that the part of the lines perpendicular to the axis of x which satisfies the equation is contained between the limits $\pm\frac{1}{4}\pi$. The following calculation will shew that these limits are erroneous. The former part of the investigation has been taken from a paper by Professor Newman, published in No. xv. of this *Journal.*

Since the series is convergent, an even or odd number of terms will, in the limit, give the same result. Let therefore

$$y = \cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \ldots - \frac{1}{4n-1}\cos(4n-1)x,$$

where n is an infinitely great integer.

$$\frac{dy}{dx} = -\sin x + \sin 3x - \sin 5x - \ldots + \sin(4n-1)x,$$

$$= \frac{\sin 4nx}{2\cos x};$$

$$\therefore\ y = c + \tfrac{1}{2}\int^x \frac{\sin 4nx}{\cos x}\,dx;$$

when $x = 0$, $y = 1 - \frac{1}{3} + \frac{1}{5} - \ldots = \frac{1}{4}\pi$,

$$\therefore\ \tfrac{1}{4}\pi = c + \tfrac{1}{2}\int^{0} \frac{\sin 4nx}{\cos x}\,dx,$$

$$y - \tfrac{1}{4}\pi = \tfrac{1}{2}\int_0^{x} \frac{\sin 4nx}{\cos x}\,dx\,;$$

For x write $\frac{1}{2}\pi - \dfrac{u}{4n}$,

$$y - \tfrac{1}{4}\pi = -\tfrac{1}{2}\int_{2n\pi}^{4n(\frac{1}{2}\pi - x)} \frac{\sin(2n\pi - u)}{\cos\left(\dfrac{\pi}{2} - \dfrac{u}{4n}\right)}\,\frac{du}{4n},$$

$$= -\tfrac{1}{2}\int_{4n(\frac{1}{2}\pi - x)}^{2n\pi} \frac{\sin u}{\sin\dfrac{u}{4n}}\,\frac{du}{4n}.$$

It is easily seen that the only values of u which affect the value of the integral are those which make $\dfrac{u}{4n}$ very small; in that case $\dfrac{1}{4n}\left(\sin\dfrac{u}{4n}\right)^{-1}$ becomes u^{-1}. Hence

$$y = \tfrac{1}{4}\pi - \tfrac{1}{2}\int_{4n(\frac{1}{2}\pi - x)}^{\infty} \frac{\sin u}{u}\,du.$$

When x differs by a finite quantity from $\frac{1}{2}\pi$, the lower as well as the upper limit is infinite, and the integral vanishes. Thus the general value of y, for values of x less than $\frac{1}{2}\pi$ is $\frac{1}{4}\pi$. When $x = \frac{1}{2}\pi$ exactly, the integral becomes $\int_0^{\infty} \dfrac{\sin u}{u}\,du$, which is known to be equal to $\frac{1}{2}\pi$; and in this case y vanishes. When, however, $4n(\frac{1}{2}\pi - x)$ is a finite quantity, that is, when x differs from $\frac{1}{2}\pi$ by an infinitesimal comparable with $\dfrac{1}{n}$, to each value of x corresponds a certain value of y.

To investigate the variations of these values, we will consider the curve $uv = \sin u$, u and v being abscissa and ordinate. Its form will be as in figure 1, where the maximum positive and negative values of v successively decrease.

The integral $\int_{u_1}^{\infty} \dfrac{\sin u}{u}\,du$ will represent the sum of the areas included between the axis of u and the successive loops of this curve from $u = u_1$ to $u = \infty$, the areas below the axis

being accounted negative. When the area begins at the origin, we know that it is equal to $\frac{1}{2}\pi$. If the point from which it is reckoned move from O towards π, the area gradually decreases, vanishes at some point M, then becomes negative, has its maximum negative value at π, thence increases to a smaller positive maximum at 2π, whence it verges to a still smaller negative maximum at 3π, and so on, being alternately positive and negative, the successive maximums converging rapidly to the value 0.

From this we gather that the locus of the equation

$$y = \cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \ldots.$$

is that represented in figure 2, the variations however near $\frac{1}{2}\pi$ being contained within infinitesimal limits of variation of x.

This shews that the assertion that when $x = \frac{1}{2}\pi$, y has any value between $\pm\frac{1}{4}\pi$, is incorrect; the truth being that, when x differs infinitesimally from $\frac{1}{2}\pi$, the values of y vary, not between $\pm\frac{1}{4}\pi$, but between limits numerically greater than these, viz.

$$\pm\left\{\tfrac{1}{4}\pi - \tfrac{1}{2}\left(\text{the greatest negative value of} \int_u^\infty \frac{\sin u}{u}\,du\right)\right\},$$

or, which amounts to the same, $\pm\int_0^\pi \frac{\sin u}{u}\,du$.

It may be interesting to examine the nature of the curve defined by the preceding equation, supposing the number of terms to be finite. We must here make a distinction between the cases in which the number of terms is even and odd.

First suppose the number to be even and equal to $2m$,

$$y = \cos x - \tfrac{1}{3}\cos 3x + \ldots. - \frac{1}{4m-1}\cos(4m-1)x,$$

$$\frac{dy}{dx} = \frac{\sin 4mx}{2\cos x},$$

$$\frac{d^2y}{dx^2} = \frac{2m\cos 4mx}{\cos x} + \frac{\sin 4mx \sin x}{2\cos^2 x}.$$

Whenever $\sin 4mx = 0$, *i.e.* whenever $x = \dfrac{r\pi}{4m}$, there will be a singular point, at which y will be a minimum when r is even, and a maximum when r is odd.

At these points $\qquad \dfrac{d^2y}{dx^2} = \pm\dfrac{2m}{\cos x}.$

Fig. 1.

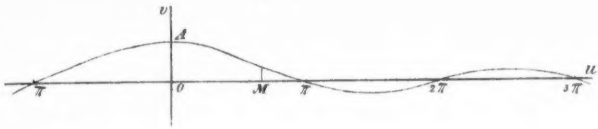


Fig. 2.

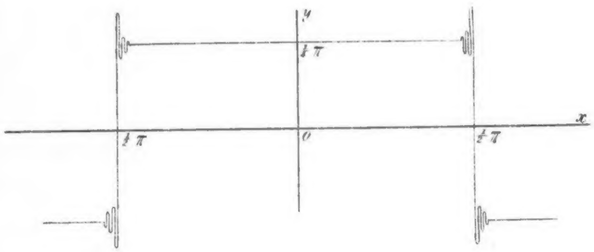


Fig. 3.

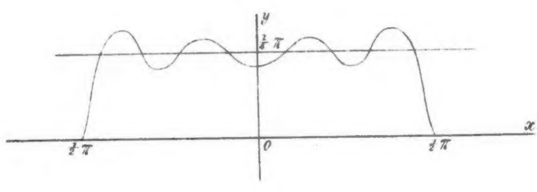
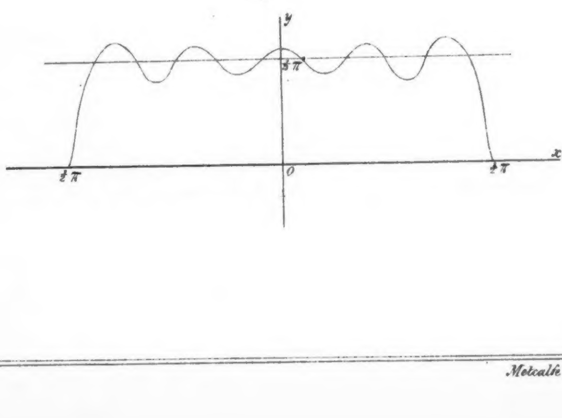


Fig. 4.



Metcalfe & Palmer, litho.

Hence the curvature at the singular points farther from the axis of y is greater than at those nearer.

When $x = 0$, y is a minimum, being less than the mean value $\frac{1}{4}\pi$. The minimums will all be less and the maximums greater than this mean value.

When $m = 2$, the curve is that represented by fig. 3.

If the number of terms be odd and equal to $2m + 1$,

$$y = \cos x - \tfrac{1}{3}\cos 3x + \ldots + \frac{1}{4m+1}\cos(4m+1)x,$$

$$\frac{dy}{dx} = -\frac{\sin(4m+2)x}{2\cos x},$$

$$\frac{d^2y}{dx^2} = -\frac{(2m+1)\cos(4m+2)x}{2\cos x} - \frac{\sin(4m+2)x\sin x}{2\cos^2 x}.$$

Whenever $\sin(4m+2)x = 0$, *i.e.* whenever $x = \dfrac{r\pi}{4m+2}$, there will be a maximum or minimum value of y according as r is even or odd. Hence when $x = 0$, y is a maximum. As in the former case, the minimums will all be less and the maximums greater than $\frac{1}{4}\pi$, and the curvature at the singular points farther from the axis of y will be greater than at those nearer. Figure 4 represents the curve when $m = 2$. An inspection of these curves will shew how, when the number of terms is indefinitely increased, each series of curves, viz. that wherein the number is even and that in which it is odd, converges to the same limiting form.

A similar investigation of the equation

$$y = \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \ldots\ldots$$

would lead to an analogous result.

Trinity College, April 6, 1848.

ON THE APPLICATION OF A SYMBOL OF DISCONTINUITY TO QUESTIONS OF MAXIMA AND MINIMA.

By William Walton.

There are many instances of problems of maxima and minima, in which the ordinary rules entirely fail in consequence of the variations of sign in the component functions of the quantity under consideration. Suppose, for instance, that $v_1, v_2, v_3, \ldots$ are any assigned functions of x, and that

our object is to ascertain the value of x corresponding to the maxima or minima of the expression

$$\pm v_1 \pm v_2 \pm v_3 + \ldots.$$

the + or − sign being prefixed to each of the functions accordingly as it is positive or negative under all conceivable variations of x: or suppose the expression to be

$$\pm v_1 \mp v_2 \pm v_3 \mp \ldots,$$

the + sign being prefixed to $v_1, v_3, v_5 \ldots.$ when positive and the − sign when negative; while $v_2, v_4, v_6. \ldots$ are subjected to the − sign when positive, and the + sign when negative. It is obvious that generally in such problems the ordinary rules, applied without modification, would not determine appropriate results.

The difficulty here stated may however be obviated by the use of an ordinary symbol of discontinuity, by the aid of which all such questions of maxima and minima are brought under the dominion of analysis.

Let $\lambda_1, \lambda_2, \lambda_3, \ldots.$ be any functions of x, the signs of which are always the same as those of $v_1, v_2, v_3, \ldots.$ respectively.

Then, to take the former instance, let

$$u = \pm v_1 \pm v_2 \pm v_3 \pm \ldots.:$$

here $$\pm v_1 = \frac{1 - 0^{\lambda_1}}{1 + 0^{\lambda_1}} v_1, \quad \pm v_2 = \frac{1 - 0^{\lambda_2}}{1 + 0^{\lambda_2}} v_2,$$

hence $$u = \frac{1 - 0^{\lambda_1}}{1 + 0^{\lambda_1}} v_1 + \frac{1 - 0^{\lambda_2}}{1 + 0^{\lambda_2}} v_2 + \frac{1 - 0^{\lambda_3}}{1 + 0^{\lambda_3}} v_3 + \ldots:$$

the maxima and minima of u may then be determined by the ordinary rules.

Similarly, in the second instance,

$$u = \frac{1 - 0^{\lambda_1}}{1 + 0^{\lambda_1}} v_1 + \frac{0^{\lambda_2} - 1}{0^{\lambda_2} + 1} v_2 + \frac{1 - 0^{\lambda_3}}{1 + 0^{\lambda_3}} v_3 + \ldots.$$

The application of the idea to a few particular questions will best serve to illustrate the principles of the method.

Ex. 1. A curve is such that its ordinate is always equal to the sum of a positive quantity of the same length as the abscissa, and the square of the excess of its abscissa, affected by its algebraical sign, above unity: to find the minimum ordinate of the curve.

The equation to the curve will be

$$y = \frac{1 - 0^x}{1 + 0^x} x + (x - 1)^2.$$

Differentiating with regard to x, we get

$$\frac{dy}{dx} = \frac{1 - 0^x}{1 + 0^x} - 2x \cdot \frac{0^x \log 0}{(1 + 0^x)^2} + 2(x - 1).$$

But when x is positive, we have, by the theory of indeterminate fractions,

$$\frac{0^x \log 0}{(1 + 0^x)^2} = 0^x \log 0 = \frac{\log 0}{0^{-x}} = \frac{0^{-1}}{-x\, 0^{-x-1}} = -\frac{0^x}{0} = 0,$$

and, when x is negative,

$$\frac{0^x \log 0}{(1 + 0^x)^2} = \frac{0^{-x} \log 0}{(1 + 0^{-x})^2} = 0, \text{ by the former case:}$$

hence, under both circumstances,

$$\frac{dy}{dx} = \frac{1 - 0^x}{1 + 0^x} + 2(x - 1).$$

Putting $\frac{dy}{dx} = 0$, we have

$$x = 1 - \tfrac{1}{2}\,\frac{1 - 0^x}{1 + 0^x};$$

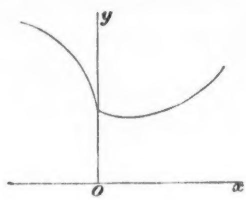
if x is positive, $\frac{1 - 0^x}{1 + 0^x} = 1$, $x = \frac{1}{2}$;

if x is negative, $\frac{1 - 0^x}{1 + 0^x} = -1$, $x = \frac{3}{2}$, which is absurd:

hence $x = \frac{1}{2}$ may give a minimum value to y; to ascertain whether this is really the case, put $x = \frac{1}{2} \mp h$; then

$$\frac{dy}{dx} = \mp 2h;$$

since therefore $\frac{dy}{dx}$ changes sign from $-$ to $+$, when x increases through the value $\frac{1}{2}$, it follows that this value of x gives a minimum value to y. The general form of the curve is here given.



Ex. 2. To find the maxima or minima of the ordinate y in a curve, when y is always equal to the excess of the magnitude of $3x$, independently of sign, above that of $(x-1)^2$.

Here $$y = 3\,\frac{1-0^x}{1+0^x}\,x - (x-1)^2:$$

proceeding as in Ex. (1), we have

$$\frac{dy}{dx} = 3\,\frac{1-0^x}{1+0^x} - 2(x-1).$$

Assuming that $\frac{dy}{dx} = 0$, we have

$$x = 1 + \tfrac{3}{2}\,\frac{1-0^x}{1+0^x}.$$

First, suppose x to be positive; then $\frac{1-0^x}{1+0^x} = 1$, and $x = \frac{5}{2}$: next, suppose x to be negative; then $\frac{1-0^x}{1+0^x} = -1$, and $x = -\frac{1}{2}$. To ascertain whether these values of x give maxima or minima of y, put first, $x = \frac{5}{2} \mp h$ in the expression for $\frac{dy}{dx}$: then

$$\frac{dy}{dx} = 5 - 2(\tfrac{5}{2} \mp h) = \pm 2h;$$

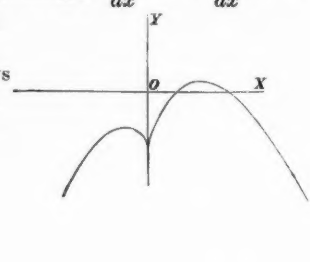
hence, as x increases through $\frac{5}{2}$, $\frac{dy}{dx}$ changes sign from + to −; therefore, $x = \frac{5}{2}$ makes y a maximum. Next, put $x = -\frac{1}{2} \mp h$; then

$$\frac{dy}{dx} = -3 + 2 - 2(-\tfrac{1}{2} \mp h) = \pm 2h;$$

hence, also $x = -\frac{1}{2}$ gives to y a maximum value.

There is still however another value of x, which makes y a minimum, which does not satisfy the condition $\frac{dy}{dx} = 0$; this value of x is zero. Putting in fact, first $x = -0$, and then $x = +0$, we have correspondingly, $\frac{dy}{dx} = -1$, $\frac{dy}{dx} = +5$; hence, $x = 0$ makes y a minimum.

The annexed diagram shews the general form of the curve.



Ex. 3. Any number of points being taken in the same straight line, to determine the point the sum of the distances of which from the given points is a minimum.

Let $c_1, c_2, c_3, \ldots$ be the distances of the proposed points from an origin taken in their line, and let x be the distance of the required point from this origin. The object of the problem is to determine the minimum value of the expression

$$u = (x \sim c_1) + (x \sim c_2) + (x \sim c_3) + \ldots$$

$$= \Sigma(x \sim c) = \Sigma\left\{\frac{1 - 0^{x-c}}{1 + 0^{x-c}}(x - c)\right\}:$$

differentiating and proceeding as in the two previous examples, we have,

$$\frac{du}{dx} = \Sigma\left\{\frac{1 - 0^{x-c}}{1 + 0^{x-c}}\right\}.$$

Now, from the expression for $\frac{du}{dx}$, it is evident that, as x keeps increasing, no change can take place in the sign of $\frac{du}{dx}$ except when x increases through some of the values $c_1, c_2, c_3, \ldots$ Suppose that there are m points on the side of a point A of the system, nearer to the origin, and n on the other; then the value of $\frac{du}{dx}$, as x increases through the abscissa of the point A, will change from $m - n - 1$ to $m - n + 1$. Suppose that $m - n - 1$ is negative, and $m - n + 1$ positive; these conditions are evidently impossible unless $m = n$, and are then satisfied; hence, if the number of points is odd, the middle point of the system is the point required.

Suppose however the number of the points of the system to be even; and let us direct our attention to two consecutive ones A and B. Let m, n, be the numbers of points on each side of these two. Then, when the moving point passes from one side of these two points to the other, $\frac{du}{dx}$ changes its value from $m - n - 2$ to $m - n + 2$. Let $m - n - 2$ be negative, and $m - n + 2$ positive; then, m and n being even, m must be equal to n. Hence, observing that $\frac{du}{dx}$ retains the same value $m - n = 0$, while the moving point lies between A and B, we see that the problem is in this case indeterminate, the required position of the moving point being any where between A and B.

Cambridge, Feb. 16, 1848.

NOTE ON THE PROBLEM OF FALLING BODIES AS AFFECTED BY THE EARTH'S ROTATION.

(To the Editor of the Cambridge and Dublin Mathematical Journal.)

The following solution of a common problem is not presented to your readers as anything new or recondite, but as an example of the advantage of solving such problems by reference to known properties, rather than by setting out from the general equations of motion. For young mathematicians, up to a certain point of their progress, such a lesson is not without its use, and if on this ground you think it worthy of insertion in your *Journal* I shall be glad to see it there.

Trinity College, Feb. 5, 1848.

Problem. A body falls from apparent rest towards the earth: to shew that there will be a deviation from the vertical line towards the east, and towards the equator; and to find approximately the value of the deviation in each direction.

The centre of the earth may be considered to be at rest, and the apparent motion only need be considered. The body falling from apparent rest is really projected due east with the velocity of the earth's rotation, and, after the projection, is acted upon by the force of gravity, which (above the earth's surface) varies inversely as the square of the distance from the centre. Therefore the body will describe an ellipse, the plane of which will pass through the earth's centre, and will consequently cut the earth's surface in a great circle passing east and west through the foot of the original vertical. And this great circle touches at that point the parallel of latitude which the foot of the vertical describes, and afterwards deviates towards the equator. Hence the body wherever it meets the earth's surface, will do so between the parallel of latitude and the equator, or in other words will deviate towards the equator.

Also it will deviate towards the east; for since the plane of the ellipse passes east and west, the body in it would, if the radius and angular velocity had remained the same, have advanced to the east through the same angle as the vertical. But during the fall the radius is shortened, and therefore the angle through which the body advances will be greater than it would have been on that supposition. Hence the body will deviate towards the east.

To find the amount of deviation. Let c be the distance from the earth's centre of A, of the point from which the body falls; ω the angular velocity of the earth, λ the latitude of A, g the force of gravity at A, and t the duration of the fall.

If a be the semiaxis major of the ellipse described by the body, e its excentricity, the radius of curvature at A is $a(1-e^2)$. The velocity at the point A is $c \cos \lambda\, \omega$, and therefore

$$g\,a\,(1-e^2) = c^2 \cos^2 \lambda\, \omega,$$

2

whence by substituting for c its value $a(1+e)$,

$$g\,a(1-e^2) = a^2(1+e)^2 \cos^2 \lambda\, \omega^2,$$

and
$$\frac{g}{a\omega^2 \cos^2\lambda} = \frac{1+e}{1-e},$$

therefore
$$e = \frac{g - a\omega^2 \cos^2\lambda}{g + a\omega^2 \cos^2\lambda}.$$

The polar equation of the ellipse is

$$r = \frac{a(1-e^2)}{1 - e\cos\theta},$$

r being the radius vector, and θ the angle which r makes with the vertical from A. When θ is very small as in our problem,

$$r = a(1+e)\left(1 - \frac{e}{1-e}\,\frac{\theta^2}{2}\right),$$

and the area of the portion of the ellipse between the original position of r, and that in which it is inclined to the major axis at an angle θ

$$= \tfrac{1}{2}\int_0^\theta r^2\, d\theta = \tfrac{1}{2} a^2 (1+e)^2 \left(\theta - \frac{e}{1-e}\,\frac{\theta^3}{3}\right),$$

But this area measures the time and is equal to $\frac{1}{2}c^2 \cos\lambda\, \omega\, t$. Hence

$$\omega t \cos\lambda = \theta - \frac{e}{1-e}\,\frac{\theta^3}{3},$$

or, approximately, $\omega t \cos\lambda = \theta$.

We have therefore for the eastern deviation $c\theta - c\cos\lambda\, \omega\, t$, the value

$$\frac{ce}{1-e}\,\frac{\theta^3}{3} = \frac{ce}{1-e}\,\frac{\omega^3 t^3 \cos^3\lambda}{3}$$

$$= \frac{a(1+e)e}{1-e}\,\frac{\omega^3 t^3 \cos^3\lambda}{3} = \frac{g - a\omega^2\cos^2\lambda}{g + a\omega^2\cos^2\lambda}\,\frac{g\cos\lambda\,\omega\, t^3}{3}.$$

In the actual case, $a\omega^2$ is small compared with g, and hence

$$\text{eastern deviation} = \frac{g\cos\lambda\,\omega\, t^3}{3} \text{ nearly.}$$

For example, when $t = 10$ seconds, the deviation is about 1 foot.

Again, it is evident that the great circle, which lies in the plane of the ellipse, and the parallel of latitude in which the foot of the vertical moves, have a common tangent at the point where they touch, and that after the time t each circle deviates from this tangent by a space $= \frac{(\text{arc})^2}{2 \text{ radius}}$. In the small circle, arc $= c \cos\lambda\, \theta$, radius $= c \cos \lambda$. Hence deviation from the tangent $= \frac{c \cos\lambda\, \theta^2}{2}$. In the great circle, arc $= c \cos\lambda\, \theta$, radius $= c$. Hence deviation $= \frac{c \cos^2\lambda\, \theta^2}{2}$. And these two deviations are lines which make an angle λ, for they are both perpendicular to the tangent which is the intersection of the planes of the great and small circle, in which they respectively lie, and these planes make an angle λ. Hence the deviation towards the equator, which is the third side of the triangle, is

$$\frac{c\theta^2}{2} \sqrt{(\cos^2 \lambda + \cos^4 \lambda - 2 \cos^4 \lambda)}$$

$$= \frac{c\theta^2}{2} \sin \lambda \cos \lambda = \frac{c\omega^2 t^2}{2} \sin \lambda \cos^3 \lambda.$$

[The greatest value of this expression is when $\lambda = 30°$, in which case if $t = 10$ seconds, the deviation will be about $\frac{1}{8}$ foot. In latitute 45° when $t = 18$ seconds, the deviation is about $\frac{1}{8}$ foot. Since $g = \frac{\mu}{c^2}$, μ being a constant depending on the earth's attraction, the expression for the eastern deviation may be

$$\frac{\mu \cos \lambda\, \omega t^3}{c^2},$$

from which we see, as might be expected *a priori*, that the increase in the eastern deviation is less rapid as the height from which the body falls is greater, for t^3 will not increase near so rapidly as c^2. On the other hand the deviation towards the equator increases more rapidly, the greater the height is. With regard to the changes owing to variation in the latitude, we see that the eastern deviation decreases from the equator to the poles, whereas the deviation towards the equator increases from $\lambda = 0$, to latitude 30°, and then decreases to the poles.—W. S.]

NOTES ON HYDRODYNAMICS.

IV. DEMONSTRATION OF A FUNDAMENTAL THEOREM.

By G. G. Stokes.

Theorem. Let the accelerating forces X, Y, Z, acting on the fluid, be such that $Xdx + Ydy + Zdz$ is the exact differential dV of a function of the coordinates. The function V may also contain the time t explicitly, but the differential is taken on the supposition that t is constant. Suppose the fluid to be either homogeneous and incompressible, or homogeneous and elastic, and of the same temperature throughout, except in so far as the temperature is altered by sudden condensation or rarefaction, so that the pressure is a function of the density. Then if, either for the whole fluid mass, or for a certain portion of it, the motion is at one instant such that $udx + vdy + wdz$ is an exact differential, that expression will always remain an exact differential, in the first case throughout the whole mass, in the second case throughout the portion considered, a portion which will in general continually change its position in space as the motion goes on. In particular, the proposition is true when the motion begins from rest.

Two demonstrations of this important theorem will here be given. The first is taken from a memoir by M. Cauchy, "Sur la Théorie de Ondes" (*Savans Etrangers*, tom. I. p. 40). M. Cauchy has obtained three first integrals of the equations of motion for the case in which $Xdx + Ydy + Zdz$ is an exact differential, and in which the pressure is a function of the density; a case which embraces almost all the problems of any interest in this subject. M. Cauchy, it is true, has only considered an incompressible fluid, in accordance with the problem he had in hand, but his method applies to the more general case in which the pressure is a function of the density. The theorem considered follows as a particular consequence from M. Cauchy's integrals. As however the equations employed in obtaining these integrals are rather long, and the integrals themselves do not seem to lead to any result of much interest except the theorem enunciated at the beginning of this article, I have given another demonstration of the theorem, which is taken from the *Cambridge Philosophical Transactions* (vol. VIII. p. 307). A new proof of the theorem for the case of an incompressible fluid will be given by Professor Thomson in this *Journal*.

First Demonstration. Let the time t and the initial coordinates a, b, c be taken for the independent variables; and let $\int \frac{dp}{\rho} = P$, p being by hypothesis a function of ρ. Since we have, by the Differential Calculus,

$$\frac{dP}{da} = \frac{dP}{dx}\frac{dx}{da} + \frac{dP}{dy}\frac{dy}{da} + \frac{dP}{dz}\frac{dz}{da},$$

with similar equations for b and c, we get from equations (1), p. 124 (Notes on Hydrodynamics, No. III.),

$$\left.\begin{aligned}\frac{dV}{da} - \frac{dP}{da} &= \frac{d^2x}{dt^2}\frac{dx}{da} + \frac{d^2y}{dt^2}\frac{dy}{da} + \frac{d^2z}{dt^2}\frac{dz}{da}\\ \frac{dV}{db} - \frac{dP}{db} &= \frac{d^2x}{dt^2}\frac{dx}{db} + \frac{d^2y}{dt^2}\frac{dy}{db} + \frac{d^2z}{dt^2}\frac{dz}{db}\\ \frac{dV}{dc} - \frac{dP}{dc} &= \frac{d^2x}{dt^2}\frac{dx}{dc} + \frac{d^2y}{dt^2}\frac{dy}{dc} + \frac{d^2z}{dt^2}\frac{dz}{dc}\end{aligned}\right\} \ldots\ldots (1).$$

In these equations $\frac{d^2x}{dt^2}$, $\frac{dx}{da}$, &c. have been written for $\frac{D^2x}{Dt^2}$, $\frac{Dx}{Da}$, &c., since the context will sufficiently explain the sense in which the differential coefficients are taken. By differentiating the first of equations (1) with respect to b, the second with respect to a, and subtracting, we get, after putting for $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ their values u, v, w,

$$\frac{d^2u}{dtdb}\frac{dx}{da} - \frac{d^2u}{dtda}\frac{dx}{db} + \frac{d^2v}{dtdb}\frac{dy}{da} - \frac{d^2v}{dtda}\frac{dy}{db} + \frac{d^2w}{dtdb}\frac{dz}{da} - \frac{d^2w}{dtda}\frac{dz}{db} = 0 \qquad \ldots\ldots\ldots\ldots (2).$$

By treating the second and third, and then the third and first of equations (1) as the first and second have been treated, we should get two more equations, which with (2) would form a symmetrical system. Now it is easily seen, on taking account of the equations $\frac{du}{dt} = u$, &c., that the first side of (2) is the differential coefficient with respect to t of

$$\frac{du}{db}\frac{dx}{da} - \frac{du}{da}\frac{dx}{db} + \frac{dv}{db}\frac{dy}{da} - \frac{dv}{da}\frac{dy}{db} + \frac{dw}{db}\frac{dz}{da} - \frac{dw}{da}\frac{dz}{db} \ldots (3),$$

the differential coefficient in question being of course of the

kind denoted by D in No. III. of these Notes. Hence the expression (3) is constant for the same particle. Let u_0, v_0, w_0 be the initial velocities of the particle which at the time t is situated at the point (x, y, z); then if we observe that $x = a$, $y = b$, $z = c$, when $t = 0$, we shall get from (2) and the two other equations of that system,

$$\left.\begin{array}{l}
\dfrac{du}{db}\dfrac{dx}{da} - \dfrac{du}{da}\dfrac{dx}{db} + \dfrac{dv}{db}\dfrac{dy}{da} - \dfrac{dv}{da}\dfrac{dy}{db} + \dfrac{dw}{db}\dfrac{dz}{da} - \dfrac{dw}{da}\dfrac{dz}{db} = \dfrac{du_0}{db} - \dfrac{dv_0}{da} \\
\dfrac{du}{dc}\dfrac{dx}{db} - \dfrac{du}{db}\dfrac{dx}{dc} + \dfrac{dv}{dc}\dfrac{dy}{db} - \dfrac{dv}{db}\dfrac{dy}{dc} + \dfrac{dw}{dc}\dfrac{dz}{db} - \dfrac{dw}{db}\dfrac{dz}{dc} = \dfrac{dv_0}{dc} - \dfrac{dw_0}{db} \\
\dfrac{du}{da}\dfrac{dx}{dc} - \dfrac{du}{dc}\dfrac{dx}{da} + \dfrac{dv}{da}\dfrac{dy}{dc} - \dfrac{dv}{dc}\dfrac{dy}{da} + \dfrac{dw}{da}\dfrac{dz}{dc} - \dfrac{dw}{dc}\dfrac{dz}{da} = \dfrac{dw_0}{da} - \dfrac{du_0}{dc}
\end{array}\right\} \qquad \ldots\ldots\ldots(4).$$

These are the three first integrals of the equations of motion already mentioned. If we replace the differential coefficients of u, v and w, taken with respect to a, b and c, by differential coefficients of the same quantities taken with respect to x, y and z, and differential coefficients of x, y and z taken with respect to a, b and c, the first sides of equations (4) become

$$\left.\begin{array}{l}
\left(\dfrac{du}{dy} - \dfrac{dv}{dx}\right)\left(\dfrac{dy}{db}\dfrac{dx}{da} - \dfrac{dy}{da}\dfrac{dx}{db}\right) + \left(\dfrac{dv}{dz} - \dfrac{dw}{dy}\right)\left(\dfrac{dz}{db}\dfrac{dy}{da} - \dfrac{dz}{da}\dfrac{dy}{db}\right) \\
\qquad + \left(\dfrac{dw}{dx} - \dfrac{du}{dz}\right)\left(\dfrac{dx}{db}\dfrac{dz}{da} - \dfrac{dx}{da}\dfrac{dz}{db}\right) \\
\left(\dfrac{du}{dy} - \dfrac{dv}{dx}\right)\left(\dfrac{dy}{dc}\dfrac{dx}{db} - \dfrac{dy}{db}\dfrac{dx}{dc}\right) + \left(\dfrac{dv}{dz} - \dfrac{dw}{dy}\right)\left(\dfrac{dz}{dc}\dfrac{dy}{db} - \dfrac{dz}{db}\dfrac{dy}{dc}\right) \\
\qquad + \left(\dfrac{dw}{dx} - \dfrac{du}{dz}\right)\left(\dfrac{dx}{dc}\dfrac{dz}{db} - \dfrac{dx}{db}\dfrac{dz}{dc}\right) \\
\left(\dfrac{du}{dy} - \dfrac{dv}{dx}\right)\left(\dfrac{dy}{da}\dfrac{dx}{dc} - \dfrac{dy}{dc}\dfrac{dx}{da}\right) + \left(\dfrac{dv}{dz} - \dfrac{dw}{dy}\right)\left(\dfrac{dz}{da}\dfrac{dy}{dc} - \dfrac{dz}{dc}\dfrac{dy}{da}\right) \\
\qquad + \left(\dfrac{dw}{dx} - \dfrac{du}{dz}\right)\left(\dfrac{dx}{da}\dfrac{dz}{dc} - \dfrac{dx}{dc}\dfrac{dz}{da}\right)
\end{array}\right\} \qquad \ldots\ldots\ldots\ldots(5).$$

Having put the first sides of equations (4) under the form (5), we may solve the equations, regarding

$$\frac{du}{dy} - \frac{dv}{dx}, \quad \frac{dv}{dz} - \frac{dw}{dy}, \quad \frac{dw}{dx} - \frac{du}{dz}$$

as the unknown quantities. For this purpose multiply equa-

tions (4) by $\frac{dz}{dc}$, $\frac{dz}{da}$, $\frac{dz}{db}$, and add; then the second and third unknown quantities will disappear. Again, multiply by $\frac{dx}{dc}$, $\frac{dx}{da}$, $\frac{dx}{db}$, and add; then the third and first will disappear. Lastly, multiply by $\frac{dy}{dc}$, $\frac{dy}{da}$, $\frac{dy}{db}$, and add; then the first and second will disappear. Putting for shortness

$$\frac{dx}{da}\frac{dy}{db}\frac{dz}{dc} - \frac{dx}{da}\frac{dy}{dc}\frac{dz}{db} + \frac{dx}{db}\frac{dy}{dc}\frac{dz}{da} - \frac{dx}{db}\frac{dy}{da}\frac{dz}{dc} + \frac{dx}{dc}\frac{dy}{da}\frac{dz}{db} - \frac{dx}{dc}\frac{dy}{db}\frac{dz}{da} = R \dots (6),$$

we thus get

$$\left.\begin{aligned}
\frac{du}{dy} - \frac{dv}{dx} &= \frac{1}{R}\left\{\frac{dz}{dc}\left(\frac{du_0}{db} - \frac{dv_0}{da}\right) + \frac{dz}{da}\left(\frac{dv_0}{dc} - \frac{dw_0}{db}\right) + \frac{dz}{db}\left(\frac{dw_0}{da} - \frac{du_0}{dc}\right)\right\} \\
\frac{dv}{dz} - \frac{dw}{dy} &= \frac{1}{R}\left\{\frac{dx}{dc}\left(\frac{du_0}{db} - \frac{dv_0}{da}\right) + \frac{dx}{da}\left(\frac{dv_0}{dc} - \frac{dw_0}{db}\right) + \frac{dx}{db}\left(\frac{dw_0}{da} - \frac{du_0}{db}\right)\right\} \\
\frac{dw}{dx} - \frac{du}{dz} &= \frac{1}{R}\left\{\frac{dy}{dc}\left(\frac{du_0}{db} - \frac{dv_0}{da}\right) + \frac{dy}{da}\left(\frac{dv_0}{dc} - \frac{dw_0}{db}\right) + \frac{dy}{db}\left(\frac{dw_0}{da} - \frac{du_0}{dc}\right)\right\}
\end{aligned}\right\} \dots\dots\dots\dots (7).$$

Consider the element of fluid which at first occupied the rectangular parallelepiped formed by planes drawn parallel to the coordinate planes through the points (a, b, c) and $(a + da, b + db, c + dc)$. At the time t the element occupies a space bounded by six curved surfaces, which in the limit becomes an oblique-angled parallelepiped. The coordinates of the particle which at first was situated at the point (a, b, c) are x, y, z at the time t; and the coordinates of the extremities of the three edges of the oblique-angled parallelepiped which meet in the point (x, y, z) are

$$\begin{aligned}
&x + \frac{dx}{da}\,da, \quad && y + \frac{dy}{da}\,da, \quad && z + \frac{dz}{da}\,da; \\
&x + \frac{dx}{db}\,db, && y + \frac{dy}{db}\,db, && z + \frac{dz}{db}\,db; \\
&x + \frac{dx}{dc}\,dc, && y + \frac{dy}{dc}\,dc, && z + \frac{dz}{dc}\,dc.
\end{aligned}$$

Consequently, by a formula in analytical geometry, the

volume of the element which at first was $da\,db\,dc$ is $R\,da\,db\,dc$ at the time t. Hence if ρ_0 be the initial density,

$$R = \frac{\rho_0}{\rho} \dots\dots\dots\dots\dots\dots (8).$$

From the mode in which this equation has been obtained, it is evident that it can be no other than the equation of continuity expressed in terms of a, b, c and t as independent variables, and integrated with respect to t.

The preceding equations are true independently of any particular supposition respecting the motion. If the initial motion be such that $u_0da + v_0db + w_0dc$ is an exact differential, and in particular if the motion begin from rest, we shall have

$$\frac{du_0}{db} - \frac{dv_0}{da} = 0, \qquad \frac{dv_0}{dc} - \frac{dw_0}{db} = 0, \qquad \frac{dw_0}{da} - \frac{du_0}{dc} = 0;$$

and since by (8) R cannot vanish, it follows from (7) that at any time t

$$\frac{du}{dy} - \frac{dv}{dx} = 0, \quad \frac{dv}{dz} - \frac{dw}{dy} = 0, \quad \frac{dw}{dx} - \frac{du}{dz} = 0,$$

or $u\,dx + v\,dy + w\,dz$ is an exact differential.

Since any instant may be taken for the origin of the time, and t may be either negative or positive, it is evident that for a given portion of the fluid $u\,dx + v\,dy + w\,dz$ cannot cease to be an exact differential if it is once such, and cannot become an exact differential, not having been such previously.

SECOND DEMONSTRATION. The equations of motion in their usual form are

$$\left.\begin{aligned} \frac{1}{\rho}\frac{dp}{dx} &= X - \frac{du}{dt} - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz}\text{*} \\ \frac{1}{\rho}\frac{dp}{dy} &= Y - \frac{dv}{dt} - u\frac{dv}{dx} - v\frac{dv}{dy} - w\frac{dv}{dz} \\ \frac{1}{\rho}\frac{dp}{dz} &= Z - \frac{dw}{dt} - u\frac{dw}{dx} - v\frac{dw}{dy} - w\frac{dw}{dz} \end{aligned}\right\} \dots (9).$$

* These equations were inadvertently written

$$\frac{1}{\rho}\frac{dp}{dx} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}, \text{ \&c.}$$

in No. III. of these Notes.

Differentiating the first of these equations with respect to y and the second with respect to x, subtracting, and observing that by hypothesis p is a function of ρ, and $Xdx + Ydy + Zdz$ is an exact differential, we have

$$\left(\frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}\right)\left(\frac{du}{dy} - \frac{dv}{dx}\right) + \frac{du}{dy}\frac{du}{dx} + \frac{dv}{dy}\frac{du}{dy}$$
$$+ \frac{dw}{dy}\frac{du}{dz} - \frac{du}{dx}\frac{dv}{dx} - \frac{dv}{dx}\frac{dv}{dy} - \frac{dw}{dx}\frac{dv}{dz} = 0 \ldots.(10).$$

According to the notation before employed,

$$\frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}$$

means the same as $\frac{D}{Dt}$. Let

$$\frac{dw}{dy} - \frac{dv}{dz} = 2\omega', \quad \frac{du}{dz} - \frac{dw}{dx} = 2\omega'', \quad \frac{dv}{dx} - \frac{du}{dy} = 2\omega''' \ldots(11);$$

then the last six terms of (10) become, on adding and subtracting $\frac{du}{dz}\frac{dv}{dz}$,*

$$2\frac{du}{dz}\omega' + 2\frac{dv}{dz}\omega'' - 2\left(\frac{du}{dx} + \frac{dv}{dy}\right)\omega'''.$$

We thus get from (10), and the other two equations which would be formed in a similar manner from (9),

$$\left.\begin{aligned}\frac{D\omega'''}{Dt} &= \frac{du}{dz}\omega' + \frac{dv}{dz}\omega'' - \left(\frac{du}{dx} + \frac{dv}{dy}\right)\omega''' \\ \frac{D\omega'}{Dt} &= \frac{dv}{dx}\omega'' + \frac{dw}{dx}\omega''' - \left(\frac{dv}{dy} + \frac{dw}{dz}\right)\omega' \\ \frac{D\omega''}{Dt} &= \frac{dw}{dy}\omega''' + \frac{du}{dy}\omega' - \left(\frac{dw}{dz} + \frac{du}{dx}\right)\omega''\end{aligned}\right\}\ldots(12).$$

Now the motion at any instant varying continuously from one point of the fluid to another, the coefficients of $\omega', \omega'', \omega'''$ on the second sides of equations (12) cannot become infinite. Suppose that when $t = 0$ either there is no motion, or the motion is such that $udx + vdy + wdz$ is an exact differential. This may be the case either throughout the whole fluid mass

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* $\frac{dw}{dx}\frac{dw}{dy}$ would have done as well.

or throughout a limited portion of it. Then $\omega', \omega'', \omega'''$ vanish when $t = 0$. Let L be a superior limit to the numerical values of the coefficients of $\omega', \omega'', \omega'''$ on the second sides of equations (12) from the time 0 to the time t: then evidently $\omega', \omega'', \omega'''$ cannot increase faster than if they satisfied the equations

$$\left.\begin{aligned} \frac{D\omega'''}{Dt} &= L(\omega' + \omega'' + \omega''') \\ \frac{D\omega'}{Dt} &= L(\omega' + \omega'' + \omega''') \\ \frac{D\omega''}{Dt} &= L(\omega' + \omega'' + \omega''') \end{aligned}\right\} \dots\dots\dots (13),$$

instead of (12), vanishing in this case also when $t = 0$. By integrating equations (13), and determining the arbitrary constants by the conditions that $\omega', \omega'', \omega'''$ shall vanish when $t = 0$, we should find the general values of ω', ω'', and ω''' to be zero. We need not, however, take the trouble of integrating the equations; for, putting for shortness

$$\omega' + \omega'' + \omega''' = \Omega,$$

we get, by adding together the right and left-hand sides respectively of equations (13),

$$\frac{D\Omega}{Dt} = L\Omega.$$

The integral of this equation is $\Omega = C\varepsilon^{Lt}$; and since $\Omega = 0$ when $t = 0$, $C = 0$; therefore the general value of Ω is zero. But Ω is the sum of the three quantities $\omega', \omega'', \omega'''$, which evidently cannot be negative, and therefore the general values of $\omega', \omega'', \omega'''$ are each zero. Since, then, $\omega', \omega'', \omega'''$ would have to be equal to zero, even if they satisfied equations (13), they must *a fortiori* be equal to zero in the actual case, since they satisfy equations (12), which proves the theorem enunciated.

It is evident that it is for a given mass of fluid, not for the fluid occupying a given portion of space, that the proposition is true, since equations (12) contain the differential coefficients $\dfrac{D\omega'}{Dt}$, &c. and not $\dfrac{d\omega'}{dt}$, &c. It is plain also that the same demonstration will apply to negative values of t.

If the motion should either be produced at first, or modified during its course, by impulsive pressures applied to the surface of the fluid, which of course can only be the case

when the fluid is incompressible, the proposition will still be true. In fact, the change of motion produced by impulsive pressures is merely the limit of the change of motion produced by finite pressures, when the intensity of the pressures is supposed to increase and the duration of their action to decrease indefinitely. The proposition may however be proved directly in the case of impulsive forces by using the equations of impulsive motion. If q be the impulsive pressure, u_0, v_0, w_0 the velocities just before, u, v, w the velocities just after impact, it is very easy to prove that the equations of impulsive motion are

$$\frac{1}{\rho}\frac{dq}{dx} = -(u - u_0), \quad \frac{1}{\rho}\frac{dq}{dy} = -(v - v_0), \quad \frac{1}{\rho}\frac{dq}{dz} = -(w - w_0) \ldots (14).$$

No forces appear in these equations, because finite forces disappear from equations of impulsive motion, and there are no forces which bear to finite forces, like gravity, acting all over the mass, the same relation that impulsive bear to finite pressures applied at the surface; and the impulsive pressures applied at the surface will appear, not in the general equations which hold good throughout the mass, but in the particular equations which have to be satisfied at the surface. The equations (14) are applicable to a heterogeneous, as well as to a homogeneous liquid. They must be combined with the equation of continuity of a liquid, (equation (6), p. 286 of the preceding volume.) In the case under consideration, however, ρ is constant; and therefore from (14)

$$(u - u_0)\, dx + (v - v_1)\, dy + (w - w_0)\, dz$$

is an exact differential $d\left(-\frac{q}{\rho}\right)$; and therefore if u_0, v_0, w_0 be zero, or if they be such that $u_0 dx + v_0 dy + w_0 dz$ is an exact differential $d\phi_0$, $udx + vdy + wdz$ will also be an exact differential $d\left(\phi_0 - \frac{q}{\rho}\right)$.

When $udx + vdy + wdz$ is an exact differential $d\phi$, the expression for dP obtained from equations (9) is immediately integrable, and we get

$$P = V - \frac{d\phi}{dt} - \frac{1}{2}\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right\} \ldots . (15),$$

supposing the arbitrary function of t introduced by integration to be included in ϕ.

M. Cauchy's proof of the theorem just considered does not seem to have attracted the attention which it deserves. It does not even appear to have been present to Poisson's mind when he wrote his *Traité de Mecanique*. The demonstration which Poisson has given* is in fact liable to serious objections.† Poisson indeed was not satisfied as to the generality of the theorem. It is not easy to understand the objections which he has raised,‡ which after all do not apply to M. Cauchy's demonstration, in which no expansions are employed. As Poisson gives no hint where to find the "examples" in which he says the theorem fails, if indeed he ever published them, we are left to conjecture. In speaking of the developments of u, v, w in infinite series of exponentials or circular functions, suited to particular problems, by which all the equations of the problem are satisfied, he remarks that one special character of such expansions is, not always to satisfy the equations which are deduced from those of motion by new differentiations. It is true that the equations which would *apparently* be obtained by differentiation would not always be satisfied; for the differential coefficients of the expanded functions cannot in general be obtained by direct differentiation, that is by differentiating under the sign of summation, but must be got from formulæ applicable to the particular expansions.‖ Poisson appears to have met with some contradiction, from whence he concluded that the theorem was not universally true, the contradiction probably having arisen from his having differentiated under the sign of summation in a case in which it is not allowable to do so.

It has been objected to the application of the theorem proved in this note to the case in which the motion begins from rest, that we are not at liberty to call $udx + vdy + wdz$ an exact differential when u, v, and w vanish with t, unless it be proved that if u_1, v_1, w_1 be the results obtained by dividing u, v, w by the lowest power of t occurring as a factor in u, v, w, and then putting $t = 0$, $u_1dx + v_1dy + w_1dz$ is an exact differential. Whether we call $udx + vdy + wdz$ in all cases an exact differential when u, v, and w vanish, is a matter of definition, although reasons might be assigned which would induce us to allow of the application of the term in all such cases: the demonstration of the theorem is

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* *Traité de Mecanique*, tom. II. p. 688 (2nd edition).

† See *Cambridge Philosophical Transactions*, vol. VIII. p. 305.

‡ *Traité de Mecanique*, tom. II. p. 690.

‖ See a paper "On the Critical Values of the sums of Periodic Series," *Cambridge Philosophical Transactions*, vol. VIII. Part 5.

not at all affected. Indeed, in enunciating and demonstrating the theorem there is no occasion to employ the term *exact differential* at all. The theorem might have been enunciated as follows. If the three quantities $\frac{du}{dy} - \frac{dv}{dx}$ &c. are numerically equal to zero when $t = 0$, they will remain numerically equal to zero throughout the motion. This theorem having been established, it follows as a result that when u, v, and w vanish with t, $u_1dx + v_1dy + w_1dz$ is an exact differential.

The theorem has been shewn to be a rigorous consequence of the hypothesis of the absence of all tangential force in fluids in motion. It now becomes a question, How far is the theorem *practically* true, or nearly true; or in what cases would it lead to results altogether at variance with observation?

As a general rule it may be answered that the theorem will lead to results nearly agreeing with observation when the motion of the particles which are moving is continually beginning from rest, or nearly from rest, or is as good as if it were continually beginning from rest; while the theorem will practically fail when the velocity of a given particle, or rather its velocity relatively to other particles, takes place for a long continuance in one direction.

Thus, when a wave of sound is propagated through air, a new set of particles is continually coming into motion; or the motion, considered with reference to the individual particles, is continually beginning from rest. When a wave is propagated along the surface of water, although the motion of the water at a distance from the wave is not mathematically zero, it is insensible, so that the set of particles which have got any sensible motion is continually changing. When a series of waves of sound is propagated in air, as for example the series of waves coming from a musical instrument, or when a series of waves is propagated along the surface of water, it is true that the motion is not continually beginning from rest, but it is as good as if it were continually beginning from rest. For if at any instant the disturbing cause were to cease for a little, and then go on again, the particles would be reduced to rest, or nearly to rest, when the first series of waves had passed over them, and they would begin to move afresh when the second series reached them. Again, in the case of the simultaneous small oscillations of solids and fluids, when the forward and backward oscillations are alike, equal velocities in opposite directions are continually impressed on the particles at intervals of time separated by

half the time of a complete oscillation. In such cases the theorem would generally lead to results agreeing nearly with observation.

If however water coming from a reservoir where it was sensibly at rest were to flow down a long canal, or through a long pipe, the tendency of friction being always the same way, the motion would soon altogether differ from one for which $udx + vdy + wdz$ was an exact differential. The same would be the case when a solid moves continually onwards in a fluid. Even in the case of an oscillating solid, when the forward and backward oscillations are not similar, as for example when a cone oscillates in the direction of its axis, it may be conceived that the tendency of friction to alter the motion of the fluid in the forward oscillation may not be compensated in the backward oscillation; so that, even if the internal friction be very small, the motion of the fluid after several oscillations may differ widely from what it would have been had there been absolutely no friction. I do not expect that there would be this wide difference; but still the actual motion would probably not agree so well with the theoretical, as in those cases in which the forward and backward oscillations are alike. By the theoretical motion is of course meant that which would be obtained from the common theory, in which friction is not taken into account.

It appears from experiments on pendulums that the effect of the internal friction in air and other gases is greater than might have been anticipated. In Dubuat's experiments on spheres oscillating in air the spheres were large, and the alteration in the time of oscillation due to the resistance of the air, as determined by his experiments, agrees very nearly with the result obtained from the common theory. Other philosophers, however, having operated on smaller spheres, have found a considerable discrepancy, which is so much the greater as the sphere employed is smaller. It appears, moreover, from the experiments of Colonel Sabine, that the resistance depends materially upon the nature of the gas. Thus it is much greater, in proportion to the density, in hydrogen than in air.

ON SYMBOLICAL GEOMETRY.

By Sir William Rowan Hamilton.

[*Continued from* p. 84.]

On Elliptic Cones, and on their Osculating Cones of Revolution.

33. With the same significations of a', $a^{\backprime}$, c, c', d'', and e, as symbols of certain straight lines, connected with a given cyclic cone, as in the last article of this Essay; and with the same use of the sign I, as the characteristic of the *index* of the vector part of any geometrical fraction in general; if we now write

$$f = \mathrm{I}\frac{a'}{c'}; \quad g = \mathrm{I}\frac{d''}{a'}; \quad h = \mathrm{I}\frac{d''}{c} = \mathrm{I}\frac{c}{c'} \ldots . (241);$$

$$i = \mathrm{I}\frac{a^{\backprime}}{c'}; \quad k = \mathrm{I}\frac{d''}{a^{\backprime}}; \quad l = \frac{h}{c}\,a^{\backprime} \parallel \mathrm{I}\frac{a'}{a^{\backprime}} \ldots . (242);$$

we shall thus form symbols for certain other straight lines, f, g, h, and i, k, l, which may be conceived to be all drawn from the same common origin as the former lines, namely from the vertex of the cyclic cone. And these new lines will be found to be connected with *another* cone, which may be called an *elliptic* cone;* namely the cone which is *normal, supplementary*, or *reciprocal* to the former *cyclic* cone. They may also be employed to assist in the determination of the *cone of revolution*, which *osculates* along a given side to this new or elliptic cone; as will be seen by the following investigation.

34. The lines f and g being, as is shewn by their expressions (241), perpendicular respectively to the planes $a'c'$ and $a'd''$, which were the two cyclic planes of the former or cyclic cone, are themselves the two *cyclic normals* of that cone; and because the line h is, by the same system of expressions (241), perpendicular to the plane $c'd''$ which touches that

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* The methods of the present Symbolical Geometry might here be employed to prove that the *normal cone*, here called *elliptic*, from its connexion with its two focal lines, is itself *another cyclic cone;* being cut in circles by two sets of planes, which are perpendicular respectively to the two focal lines of the former cone. But it may be sufficient thus to have alluded to this well-known theorem, which it is not necessary for our present purpose to employ. There is even a convenience in retaining, for awhile, the two contrasted designations of *cyclic* and *elliptic*, for these two reciprocal cones, to mark more strongly the difference of the modes in which they here present themselves to our view.

cyclic cone along the side c, it is the variable normal of that former cone: or this new line h is the *side* of the new or normal cone, which *corresponds* to that old side c. The inclinations of h to f and g, respectively, are given by the following equations, which are consequences of the same expressions (241):

$$\left.\begin{array}{l}\angle(\mathrm{f, h}) = \angle(\mathrm{a', c', c}) = \angle(\mathrm{a', c', d''})\\ \angle(\mathrm{h, g}) = \angle(\mathrm{a', d'', c}) = \angle(\mathrm{a', d'', c'})\end{array}\right\}\ldots\ldots(243);$$

and we have seen, in article 24, that for the cyclic cone an equation which may now be thus written holds good:

$$\angle(\mathrm{a', c', d''}) + \angle(\mathrm{a', d'', c'}) = 2a\ldots\ldots(244);$$

where a is a constant angle: therefore for the cone of normals to that cyclic cone, the following other equation is satisfied:

$$\angle(\mathrm{f, h}) + \angle(\mathrm{h, g}) = 2a\ldots\ldots\ldots(245);$$

a being here the same constant as before. The sum of the inclinations of the variable side h of the new or *elliptic* cone to the two fixed lines f and g is therefore constant; in consequence of which known property, these two fixed lines are called the *focal lines* of the elliptic cone. And we see that these two *focal lines* f, g, of the *normal* cone, coincide, respectively, in their directions, with the two *cyclic normals* (or with the normals to the two cyclic planes) of the *original* cone: which is otherwise known to be true.

35. Another important and well-known property of the elliptic cone may be proved anew by observing that the expressions (241) give

$$\left.\begin{array}{l}\angle(\mathrm{f, h, c}) = \angle(\mathrm{f, h, c'}) - \angle(\mathrm{c, h, c'}) = \tfrac{1}{2}\pi - \angle(\mathrm{c, c'})\\ \angle(\mathrm{c, h, g}) = \angle(\mathrm{d'', h, g}) - \angle(\mathrm{d'', h, c}) = \tfrac{1}{2}\pi - \angle(\mathrm{d'', c})\end{array}\right\}\ldots(246);$$

and that we have, by (190),

$$\angle(\mathrm{d'', c}) = \angle(\mathrm{c, c'})\ldots\ldots\ldots\ldots(247);$$

for thus we see that

$$\angle(\mathrm{f, h, c}) = \angle(\mathrm{c, h, g})\ldots\ldots\ldots(248);$$

that is to say, the lateral normal plane hc to the reciprocal or elliptic cone (which is at the same time the lateral normal plane of the original or cyclic cone) bisects the dihedral angle $\angle(\mathrm{f, h, g})$, comprised between the two *vector planes*, fh, hg, which connect the side h of the elliptic cone with the two focal lines f and g.

Or because the expressions (241) shew that these two vector planes, fh, hg, of the elliptic cone, are pependicular respectively to the two *traces* c′ and d″ of the tangent plane to the cyclic cone, on the two cyclic planes of that cone; which traces are, as the formula (247) expresses, inclined equally to the side of contact c of the original or cyclic cone, while that side or line c is also the normal to the reciprocal or elliptic cone; we might hence infer that the tangent plane to the latter cone is equally inclined to the two vector planes: which is another form of the same known relation.

36. The expressions (242), combined with (241), shew that the two new lines i and k, as being perpendicular respectively to the two traces c′ and d″, are contained respectively in the two vector planes, fh and hg. But each of the same two new lines, i, k, is also perpendicular to the line a‵, to which the remaining new line l is also perpendicular, as the same expressions shew; they shew too that a‵ is a line in the common lateral and normal plane ch of the two cones, while l is also contained in that plane: the plane ik therefore cuts the plane ch perpendicularly in the line l. This latter line l is also, by the same expressions, perpendicular to the line a′ (that is to the intersection of the two cyclic planes of the cyclic cone), which is perpendicular to both f and g; and therefore l can be determined, as the intersection of the common normal plane ch with the plane of the two focal lines fg; after which, by drawing through the line l, thus found, a plane ik perpendicular to ch, the lines i and k may be obtained, as the respective intersections of this last perpendicular plane with the two vector planes, fh, hg. And we see that these three new lines, i, k, l, introduced by the expressions (242), are such as to satisfy the following conditions of dihedral perpendicularity:

$$\tfrac{1}{2}\pi = \angle\,(\mathrm{h},\ \mathrm{l},\ \mathrm{i}) = \angle\,(\mathrm{k},\ \mathrm{l},\ \mathrm{h}) \ldots\ldots (249);$$

$$\tfrac{1}{2}\pi = \angle\,(\mathrm{h},\mathrm{i},\mathrm{c}') = \angle\,(\mathrm{d}'',\mathrm{k},\mathrm{h}) \ldots\ldots (250);$$

$$\tfrac{1}{2}\pi = \angle\,(\mathrm{a}`,\mathrm{c}',\mathrm{i}) = \angle\,(\mathrm{a}`,\mathrm{d}'',\mathrm{k}) \ldots\ldots (251);$$

with which we may combine the following relations:

$$\angle\,(\mathrm{f},\mathrm{h},\mathrm{i}) = \angle\,(\mathrm{k},\mathrm{h},\mathrm{g}) = 0;\quad \angle\,(\mathrm{f},\mathrm{l},\mathrm{g}) = \pi;$$
$$\angle\,(\mathrm{f},\mathrm{h},\mathrm{l}) = \angle\,(\mathrm{l},\mathrm{h},\mathrm{g}) \ldots\ldots\ldots (252).$$

37. The positions of these three lines i, k, l, being thus fully known, by means of the expressions (242), or of the corollaries which have been deduced from those expressions,

let us now consider, in connexion with them, the two formulæ of dihedral perpendicularity, (238), (240), which were given in article 32, to determine the axis e of a cone of revolution, which osculates along the side c to the given cyclic cone, and which formulæ may be thus collected:

$$\tfrac{1}{2}\pi = \angle(\mathrm{a}', \mathrm{c}', \mathrm{e}) = \angle(\mathrm{a}', \mathrm{d}'', \mathrm{e}) \ldots\ldots (253).$$

The comparison of (253) with (251) shews that the planes c′e, d″e, must coincide respectively with the planes c′i, d″k; because they are drawn like them respectively through the lines c′, d″, and are like them perpendicular respectively to the planes a′c′, a′d″; the line e must therefore be the intersection of the two planes c′i, d″k, which contain respectively the two lines i, k, and are, by (250), perpendicular to the two planes ih, kh, or (by what has been seen in the last article) to the two vector planes fh, gh. We can therefore construct the line e as the intersection of the two planes ie, ke, which are thus drawn through the lately determined lines i, k, at right angles to the two vector planes; and we may write, instead of (253), the formulæ

$$\tfrac{1}{2}\pi = \angle(\mathrm{h}, \mathrm{i}, \mathrm{e}) = \angle(\mathrm{h}, \mathrm{k}, \mathrm{e}) \ldots\ldots (254).$$

38. Again, because this line e is (by Art. 32) the axis of a cone of revolution which *osculates* to the given cyclic cone, or which touches that cone not only along the side c itself but also along another side infinitely near thereto; while, in general, the lateral normal planes of a cone of revolution all cross each other along the axis of that cone; it is clear that e must be the line along which the common lateral and normal plane ch of the two reciprocal cones is intersected by an infinitely near normal and lateral plane to the first or cyclic cone, which is also at the same time a lateral and normal plane to the second or elliptic cone; consequently *the two cones of revolution which osculate to these two reciprocal cones, along these two corresponding sides,* c and h, *have one common axis,* e. And it is evident that a similar result for a similar reason holds good, in the more general case of *any two reciprocal cones,* which have a common vertex, and of which each contains upon its surface all the normals to the other cone, *however arbitrary the form* of either cone may be; any two such cones having always *one common system of lateral and normal planes,* and *one common conical envelope of all those normal planes:* which common envelope is thus the *common conical surface of centres of curvature,* for the two reciprocal cones.

Eliminating therefore what belongs, in the present question, to the original or cyclic cone, or confining ourselves to the formulæ (245), (249), (252), (254), we are conducted to the following *construction, for determining the axis* e *of that new cone of revolution, which osculates along a given side* h *to a given elliptic cone;* this latter cone having f and g for its given focal lines, or being represented by an equation of the form (245):—*Draw, through the given side,* h, *the normal plane* hl, *bisecting the angle between the two vector planes,* fh, gh, *and meeting in the line* l *the plane* fg *of the two given focal lines; through the same line* l *draw another plane* ik, *perpendicular to the normal plane* hl, *and cutting the vector planes in two new lines,* i *and* k; *through these new lines draw two new planes,* ie, ke, *perpendicular respectively to the two vector planes,* fi, gk, *or* fh, gh: *these new planes will cross each other on the normal plane, in the sought axis* e *of the osculating cone of revolution.*

39. Or if we prefer to consider, instead of cones and planes, their intersections with a spheric surface described about the common vertex, as its centre; we then arrive at the following *spherographic construction, for finding the spherical centre of curvature of a given spherical ellipse,* at any given point of that curve, which may be regarded as being the *reciprocal* of the construction assigned at the end of the 32nd article of this essay:—Draw, from the given point H, of the ellipse, the normal arc HL, bisecting the spherical angle FHG between the two vector arcs FH, GH, and terminated at L by the arc FG which connects the two given foci, F and G; through L draw an arc of a great circle IK, perpendicular to the normal arc HL, and cutting one of the two vector arcs HF, HG, and the other of those two vector arcs prolonged, in two new points, I and K; through these two new points draw two new arcs of great circles, IE, KE, perpendicular respectively to the two vector arcs, or to the arcs HI, HK: *the two new arcs so drawn will cross each other on the normal arc* (*prolonged*), *in a point* E, *which will be the spherical centre of curvature sought,* or the pole of the small circle which osculates at the given point H to the given spherical ellipse.

And since it is obvious (on account of the spherical right angles HIE, HKE, in the construction), that the points I, K are the respective middle points of those portions of the vector arcs, or of those arcs prolonged, which are comprised within this osculating circle; so that the arc IK, which has

been seen to pass through the point L, and which crosses at that point L the arcual major axis of the ellipse (because that axis passes through both foci), is the *common bisector* of these two intercepted portions of the vector arcs, which intercepted arcs of great circles may be called (on the sphere) the two *focal chords of curvature* of the spherical ellipse; we are therefore permitted to enunciate the following *theorem*,* which is in general sufficient for the determination of the spherical centre of curvature, or pole of the osculating small circle, at any proposed point of any such ellipse:—*The great circle which bisects the two focal (and arcual) chords of curvature of any spherical ellipse, for any point of osculation, intersects the (arcual) axis major in the same point in which that axis is cut by the (arcual) normal to the ellipse, drawn at the point of osculation.*

[*To be continued.*]

ON CERTAIN POINTS IN THE THEORY OF THE CALCULUS OF VARIATIONS.

By the Rev. HARVEY GOODWIN.

THE form which the Calculus of Variations assumed under the hands of Lagrange appears to leave nothing to desire, so far as the practical application of the Calculus is concerned. If it is required to determine the relation between x and y, which will make the integral $\int V dx$, where V is a function of x and y and of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}\ldots$, taken between certain limits, a maximum or a minimum, the method adopted by Lagrange (*Leçons sur le Calcul des*

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* This theorem was proposed by the present writer, in June 1846, at the Examination for Bishop Law's Mathematical Premium, in Trinity College, Dublin; and it was shewn by him in a series of Questions on that occasion, which have since been printed in the Dublin University Calendar for 1847, (see p. LXX), among the University Examination Papers for the preceding year, that this theorem, and several others connected therewith (for example, that the trigonometric tangent of the focal half chord of curvature is the harmonic mean between the tangents of the two focal vector arcs), might be deduced by *spherical trigonometry*, from the known constancy of the sum of the two vector arcs, or focal distances, for any one spherical ellipse. But in the method employed in the present essay, no use whatever has hitherto been made of any formula of spherical or even plane trigonometry, any more than of the doctrine of coordinates.

Fonctions), and which is given in Airy's *Tracts*, is to suppose x to become $x + \delta x$ and y to become $y + \delta y$, and having found the corresponding increment of the integral to equate it to zero. The equation which results divides itself into two parts; one of which gives us the required relation between x and y, and the other the conditions to be satisfied at the limits of the integral. Now it is a fact, at first sight rather remarkable, that whether we vary x only, or y only, or both x and y, as above described, we obtain the same equation for the relation between x and y, the greater generality introduced by the variation of both variables referring only to the limits. Mr. Airy has explained, in the *Tract* above referred to, the reason why it is necessary to make the double variation, but I think that the whole theory will be exhibited in a clearer light by treating the problem as follows.

Let
$$u = \int V dx,$$
where the limits of the integral are supposed to be given by some particular conditions peculiar to each problem. And let

$$\frac{dy}{dx} = p,\ \frac{d^2y}{dx^2} = q, \ldots\ \frac{dV}{dx} = M,\ \frac{dV}{dy} = N,\ \frac{dV}{dp} = P,\ \frac{dV}{dq} = Q, \ldots$$

according to Lagrange's notation (adopted by Mr. Airy). Also, for simplicity of conception, suppose x and y to be the current coordinates in some curve, the form of which is in fact the principal thing sought. Then it is manifest that we may give the most general variation possible to the position of the point (x, y), by giving it a small tangential and a small normal displacement; call these displacements τ and ν respectively; then if δx, δy be the corresponding displacements parallel to the axes of coordinates, and ds an element of the arc of the curve,

$$\tau = \delta x \frac{dx}{ds} + \delta y \frac{dy}{ds},$$

$$\nu = \delta x \frac{dy}{ds} - \delta y \frac{dx}{ds};$$

and
$$\delta x = \tau \frac{dx}{ds} + \nu \frac{dy}{ds},$$

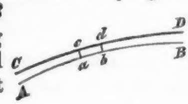
$$\delta y = \tau \frac{dy}{ds} - \nu \frac{dx}{ds}.$$

If we now proceed to find the variation of the integral in the usual manner, we obtain this result:

$$\begin{aligned}\delta u = V\left(\tau\frac{dx}{ds} + \nu\frac{dy}{ds}\right) - P\nu\frac{ds}{dx}& \\ + \frac{dQ}{dx}\nu\frac{ds}{dx} - Q\frac{d}{dx}\left(\nu\frac{ds}{dx}\right)& \\ - \frac{d^2R}{dx^2}\nu\frac{ds}{dx} + \frac{dR}{dx}\frac{d}{dx}\left(\nu\frac{ds}{dx}\right) - R\frac{d^2}{dx^2}\left(\nu\frac{ds}{dx}\right)& \\ + \&\text{c.}& \\ - \int\nu\, ds\left(N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&\text{c.}\right)& \\ = L - \int\nu U ds, \text{ suppose.}&\end{aligned}$$

This differs from the well-known expression for the variation of $\int Vdx$ merely by being given in terms of τ and ν instead of δx and δy. The first thing which strikes us in it is, that the variation τ has disappeared from every term except the first of the quantity L, and entirely from U. But the reason of this it is not difficult to see, for the equation $U = 0$ gives us the *form* of the curve sought, and it is manifest that a curve may be made to vary into another, not differing much from itself, by a normal variation only, and in fact that a tangential variation can have no effect upon the form of the curve, because if a point be displaced along the tangent or (which is the same thing) along the arc, it still remains in the same curve. The term $V\tau\dfrac{dx}{ds}$ is the variation on the supposition that $\nu = 0$, and is merely equivalent to $V\delta x$, so that by a tangential variation we obtain no term in δu depending upon the form of the curve.

The expression for δu arranges itself most conveniently in terms of the quantity νds; this seems to be because, in considering the variation as tangential and normal, we must suppose the curve to be varied into another by passing from one point to a consecutive one *along the curve,* that is, giving the variation ds, and then adding the normal variation ν. It may be remarked that νds represents an element of the area contained between the curve which we are varying and the consecutive curve; thus, suppose AB to be any curve, CD a curve differing slightly from it and supposed to be formed from it by small normal variations; let



$ab = ds$, an element of the curve, and $ac = v$, then the elementary area $acdb = vds$. Moreover, if we conceive a surface, of which the equation is

$$z = U,$$

to be described, then $\int vUds$ represents an elementary cylindrical solid bounded by the two curves AB, CD, and the surface $z = U$. If the integration is performed along the line $U = 0$ the solid vanishes.

In making u a maximum or minimum, when $u = \int Vdx$, and V involves three variables $x\, y\, z$, we obtain two equations corresponding to $U = 0$, and which we may denote by $U = 0$ and $U' = 0$. The mode by which these are ordinarily obtained corresponds to satisfying the condition of maximum or minimum for the projections of the curve of double curvature (the form of which is sought), upon two of the coordinate planes: the reason of the existence of these two equations, however, will be seen at once from what has been said; for they must necessarily depend upon a normal variation, but a curve of double curvature has a normal *plane*, not a normal line, consequently the normal variation may be resolved into two at right angles to each other, and will therefore give rise to two equations.

Similar reasoning to that which has been applied to curve lines, or to integrals in which the thing sought may be regarded as the equation of a curve line, is applicable to surfaces or to double integrals in which the thing sought is an equation of two independent variables, or the equation of a surface. For the only kind of variation which can change one surface into another is a normal variation, a variation along the tangent plane having no other effect than that of changing the position of a point in the same surface. Hence if in the double integral $\int V dx\, dy$ we put for x, $x + \delta x$; for y, $y + \delta y$; and for z, $z + \delta z$; and if $U = 0$ be the equation of the surface required, so that the direction cosines of the normal at the point $x\, y\, z$ of the surface are proportional to $\frac{dU}{dx}, \frac{dU}{dy}, \frac{dU}{dz}$ respectively, the unintegrable portion of $\delta \int V dx\, dy$ will involve only

$$\frac{dU}{dx}\delta x + \frac{dU}{dy}\delta y + \frac{dU}{dz}\delta z,$$

or the resolved part of $\delta x\, \delta y\, \delta z$ along the normal; and this will be found to be the case.

Lacroix gives certain results, which he speaks of as remarkable, the meaning of which is easily seen from what precedes.

If U be a function of x and y and their differentials, such that

$$dU = Mdx + Nd^2x + Pd^3x + Qd^4x + \dots$$
$$+ mdy + nd^2y + pd^3x + qd^4x + \dots$$

it is shewn that

$$\int\delta U = \text{an integrated part}$$
$$+ \int(M - dN + d^2P - d^3Q + \dots)\,\delta x$$
$$+ \int(m - dn + d^2p - d^3q + \dots)\,\delta y.$$

If these last two integrals be denoted by $\int\chi\delta x$ and $\int\psi\delta y$ respectively, it is shewn that χ and ψ must satisfy the following "relation remarquable,"

$$\chi dx + \psi dy = 0.$$

The meaning of this equation manifestly is this, that the sum of the resolved parts of χ and ψ along the tangent must be zero; and the reason of this is that, if it were not so, a further reduction by integration might be effected, since, as we have seen, the only unintegrable portion must be that depending upon normal variation.

The same remarks extend to the case of U being a function of three variables x, y, z, in which case the unintegrated part of $\int\delta U$ is represented by

$$\int\chi\delta x + \int\psi\delta y + \int\phi\delta z,$$

and it is shewn that χ, ψ, and ϖ must satisfy the equation

$$\chi dx + \psi dy + \phi dz = 0,$$

which equation expresses that the sum of the resolved parts of χ, ψ, and ϕ along the tangent to the curve of which x, y, and z are current coordinates is zero, or that the unintegrated portion depends upon normal variation as before.

In the case of two independent variables the unintegrable part of $\delta\int Vdxdy$ presents itself under the form

$$\int\phi\,(\delta z - p\delta x - q\delta y)\,dx\,dy.$$

It is remarked by Lacroix that if this be written under the form

$$\int(X\delta x + Y\delta y + Z\delta x)\,dx\,dy,$$

then $X\ Y\ Z$ satisfy the conditions

$$X + pZ = 0, \quad Y + qZ = 0;$$

that is, the unintegrated portion depends upon a variation in the direction of the normal to the surface of which $x\,y\,z$ are the current coordinates. This is quite analogous to the case of one independent variable, as will be seen moreover from the fact that the quantity $\delta z - p\delta x - q\delta y$ represents a variation which cannot be zero unless $\delta z = p\delta x + q\delta y$, that is, unless the variation takes place along the tangent plane. Or we may remark that

$$\delta z - p\delta x - q\delta y = \frac{1}{\frac{dU}{dz}}\left(\frac{dU}{dx}\delta x + \frac{dU}{dy}\delta y + \frac{dU}{dz}\delta z\right),$$

and therefore the above integral depends upon the normal variation $\frac{dU}{dx}\delta x + \frac{dU}{dy}\delta y + \frac{dU}{dz}\delta z$, as before observed.

The preceding remarks have been entirely devoted to the elucidation of existing formulæ by reference to certain geometrical considerations; I shall now apply the same considerations to investigate some new formulæ, which in virtue of the principles employed assume a very elegant form. As in the expression for the variation of $\int V dx$, the unintegrable portion, upon which the required relation between x and y wholly depends, does not involve the tangential variation, so conversely if we seek a relation between x and y which shall make $\delta\int V dx = 0$, we may make x and y vary, subject to the condition that the tangential portions of the variations shall vanish; that is, we may make x and y vary, subject to the condition

$$\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y = 0\,;$$

or if we have three variables, we may make them vary subject to the condition

$$\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z = 0.$$

For the sake of symmetry I shall suppose s, the arc of the curve, to be the independent variable, so that $u = \int V ds$; and shall first take the case of two variables x and y only; also I shall use the following notation:

$$\frac{dx}{ds} = x_1, \quad \frac{d^2x}{ds^2} = x_2, \quad \&c.$$

$$\frac{dy}{ds} = y_1, \quad \frac{d^2y}{ds^2} = y_2, \quad \&c.$$

$$\frac{dV}{dx} = X, \quad \frac{dV}{dx_1} = X_1, \text{ \&c.}$$

$$\frac{dV}{dy} = Y, \quad \frac{dV}{dy_1} = Y_1, \text{ \&c.}$$

So that $dV = Xdx + Ydy + X_1dx_1 + Y_1dy_1 + \ldots\ldots$ (1),

$$= (Xx_1 + Yy_1 + X_1x_2 + Y_1y_2 + \ldots.)\, ds,$$

and $\delta V = X\delta x + Y\delta y + X_1\delta x_1 + Y_1\delta y_1 + \ldots\ldots$ (2).

Then the condition to which we have subjected the variations δx and δy is

$$x_1\delta x + y_1\delta y = 0 \ldots\ldots\ldots\ldots\ldots (3).$$

Also since $x_1^2 + y_1^2 = 1 \ldots\ldots\ldots\ldots\ldots$ (4),

therefore $x_1\delta x_1 + y_1\delta y_1 = 0 \ldots\ldots\ldots\ldots\ldots$ (5),

and $x_1dx + y_1dy = ds \ldots\ldots\ldots\ldots$ (6).

Now to find the variation δu, we have

$$\delta u = \delta\int Vds = \int\{ds\delta V + V\delta(x_1dx + y_1dy)\},$$

$= \int\{ds\delta V + V(x_1d\delta x + y_1d\delta y)\}$, by (5),

$= \int\{ds\delta V - \delta x d(Vx_1) - \delta y d(Vy_1)\}$, omitting the integrated portion,

$= \int\{ds\delta V - Vds(x_2\delta x + y_2\delta y)\}$, by (3),

$$= \int ds\,\{X\delta x + Y\delta y + X_1\delta x_1 + Y_1\delta y_1 + \ldots.$$
$$+ X_n\delta x_n + Y_n\delta y_n + \ldots. - V(x_2\delta x + y_2\delta y)\}.$$

To reduce this expression we have

$$\delta x_n = \delta\,\frac{dx_{n-1}}{ds} = \delta\,\frac{dx_{n-1}}{x_1dx + y_1dy}$$

$$= \frac{d\delta x_{n-1}}{ds} - \frac{x_n}{ds}(x_1d\delta x + y_1d\delta y), \text{ by (5);}$$

similarly $\delta y_n = \dfrac{d\delta y_{n-1}}{ds} - \dfrac{y_n}{ds}(x_1d\delta x + y_1d\delta y)$;

therefore $\int ds\,(X_n\delta x_n + Y_n\delta y_n) = \int\{X_nd\delta x_{n-1} + Y_nd\delta y_{n-1}$
$- (X_nx_n + Y_ny_n)(x_1d\delta x + y_1d\delta y)\}$

$$= -\int ds\left\{\left(\frac{dX_n}{ds}\,\delta x_{n-1} + \frac{dY_n}{ds}\,\delta y_{n-1}\right) - (X_nx_n + Y_ny_n)(x_2\delta x + y_2\delta y)\right\}$$

(by integrating by parts and omitting the integrated portions)

$$= \int ds \left\{ \frac{d^2 X_n}{ds^2} \delta x_{n-2} + \frac{d^2 Y_n}{ds^2} \delta y_{n-2} \right.$$

$$\left. + \left(X_n x_n + Y_n y_n - \frac{dX_n}{ds} x_{n-1} - \frac{dY_n}{ds} y_{n-1} \right) (x_2 \delta x + y_2 \delta y) \right\}$$

in like manner,

$$= (\text{finally})\, (-1)^n \int ds \left\{ \frac{d^n X_n}{ds^n} \delta x + \frac{d^n Y_n}{ds^n} \delta y \right\}$$

$$+ \int ds \left\{ X_n x_n - \frac{dX_n}{ds} x_{n-1} + \frac{d^2 X_n}{ds^2} x_{n-2} \ldots . \right.$$

$$\left. + Y_n y_n - \frac{dY_n}{ds} y_{n-1} + \frac{d^2 Y_n}{ds^2} y_{n-2} \ldots \right\} (x_2 \delta x + y_2 \delta y).$$

By means of this formula we may write the unintegrable portion of δu under this form,

$$\delta u = \int ds \, (P \delta x + Q \delta y)$$

$$= \int ds \, \{(P + \lambda x_1) \delta x + (Q + \lambda y_1) \delta y\},$$

by (3), λ being an indeterminate multiplier, and where

$$P = -Vx_2 + X - \frac{dX_1}{ds} + \frac{d^2 X_2}{ds^2} - \&c.$$

$$+ x_2 \left\{ x_1 \left(X_1 - \frac{dX_2}{ds} + \frac{d^2 X_3}{ds^2} - \ldots . \right) \right.$$

$$+ x_2 \left(X_2 - \frac{dX_3}{ds} + \ldots . \right)$$

$$+ x_3 (X_3 - \ldots .)$$

$$+ y_1 \left(Y_1 - \frac{dY_2}{ds} + \frac{d^2 Y_3}{ds^2} - \ldots . \right)$$

$$+ y_2 \left(Y_2 - \frac{dY_3}{ds} + \ldots . \right)$$

$$\left. + y_3 (Y_3 - \ldots) \right\}$$

$$Q = -Vy_2 + Y - \frac{dY_1}{ds} + \frac{d^2 Y_2}{ds^2} - \&c.$$

$$+ y_2 \{\text{the same expression as for } P\}.$$

The condition of maximum or minimum, or $\delta u = 0$, lead

us to the two equations

$$\left.\begin{aligned} P + \lambda x_1 = 0 \\ Q + \lambda y_1 = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (7),$$

and the indeterminate multiplier λ may be expelled from these equations either by division, thus,

$$\frac{P}{x_1} = \frac{Q}{y_1} \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

or we may multiply by $x_2 y_2$ and add, and we obtain the result

$$Px_2 + Qy_2 = 0 \ldots\ldots\ldots\ldots\ldots (9).$$

It is manifest that the investigation which precedes applies *mutatis mutandis* to the case of three variables, so that we may write

$$\begin{aligned} \delta u &= \int ds \, \{P\delta x + Q\delta y + R\delta z\} \\ &= \int ds \, \{(P + \lambda x_1)\, \delta x + (Q + \lambda y_1)\, \delta y + (R + \lambda z_1)\, \delta z\}, \end{aligned}$$

where R is an expression for z similar to P for x and Q for y. And the condition $\delta u = 0$ leads us to the equations

$$\left.\begin{aligned} P + \lambda x_1 = 0 \\ Q + \lambda y_1 = 0 \\ R + \lambda z_1 = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (10),$$

which are equivalent to

$$\frac{P}{x_1} = \frac{Q}{y_1} = \frac{R}{z_1} \ldots\ldots\ldots\ldots\ldots (11).$$

There can be no doubt that the equations which have been obtained in the case of two and three variables may be extended to any number; that is, if V be a function of the variables $x, y, z, w, \ldots$ and s be another variable determined by the relation

$$dx^2 + dy^2 + dz^2 + dw^2 + \ldots. = ds^2,$$

then the equations, which give the relations between the variables necessary to make $\int V ds$ a maximum or minimum, will be

$$\frac{P}{x_1} = \frac{Q}{y_1} = \frac{R}{z_1} = \frac{S}{w_1} = \&c. \ldots\ldots\ldots\ldots (12),$$

S being the same function of w that P, Q, R are of x, y, z respectively.

This is a rather curious generalization, because the geometrical considerations which suggested the mode of proceeding

in the former case have here no existence; but the variation is nevertheless conducted subject to the condition

$$x_1\delta x + y_1\delta y + z_1\delta z + w_1\delta w + \ldots\ldots = 0,$$

an equation the meaning of which it is perhaps not so difficult to see as to enunciate in words.

It may be worth while to test the principle, which has been used in the preceding investigations, by introducing it into the ordinary method of finding the variation of $\int V dx$.

If $u = \int V dx$, we have

$$\delta u = L - \int (dy\,\delta x - dx\,\delta y)\,U;$$

or, taking into account the condition $x_1\delta x + y_1\delta y = 0$,

$$\delta u = L - \int \{(dy\,\delta x - dx\,\delta y)\,U + \lambda\,(dx\,\delta x + dy\,\delta y)\},$$

therefore $\delta u = 0$ gives us

$$Udy + \lambda dx = 0,$$

$$-\,Udx + \lambda dy = 0.$$

Multiplying these equations by dx and dy respectively, and adding, we have
$$\lambda ds^2 = 0,$$
therefore
$$\lambda = 0,$$

and the equations reduce themselves to $U = 0$, the ordinary condition.

I have before remarked that the formulæ in this subject, as given by Lagrange, are for the most part as convenient as may be for practical applications. Nevertheless, even in this respect, there are instances in which the symmetrical equations given in this paper are applicable with advantage. They are, as may be supposed, those in which the character of the problem naturally suggests the quantity s as the independent variable. I shall subjoin a few examples of the application of the formulæ.

The following is taken from the Senate-house Problems of 1847.

"The refractive index (μ) of a transparent medium varies continuously from point to point; prove by the principle of quickest propagation that the differential equations to a ray of light within the medium are

$$\frac{ds}{dx}\left\{\frac{d\mu}{dx} - \mu\,\frac{d^2x}{ds^2}\right\} = \frac{ds}{dy}\left\{\frac{d\mu}{dy} - \mu\,\frac{d^2y}{ds^2}\right\} = \frac{ds}{dz}\left\{\frac{d\mu}{dz} - \mu\,\frac{d^2z}{ds^2}\right\},$$

where the differential coefficients of μ are partial."

In this case $u = \int \mu ds$, where μ is a function of xyz only; consequently

$$P = X - Vx_2 = \frac{d\mu}{dx} - \mu \frac{d^2x}{ds^2},$$

$$Q = Y - Vy_2 = \frac{d\mu}{dy} - \mu \frac{d^2y}{ds^2},$$

$$R = Z - Vz_2 = \frac{d}{dz} - \mu \frac{d^2z}{ds^2},$$

and hence the equations (11) are coincident with those above given.

Again, let it be required to find the shortest line connecting two points on a given surface.

Let $F = 0$ be the equation of the surface; then the variations δx, δy, δz must satisfy the condition

$$\frac{dF}{dx}\delta x + \frac{dF}{dy}\delta y + \frac{dF}{dz}\delta z = 0 \quad \text{........ (13).}$$

And in this case $\qquad u = \int ds.$

Taking account of the condition (13) we must have, instead of the equations (10),

$$P + \lambda x_1 + \lambda' \frac{dF}{dx} = 0,$$

$$Q + \lambda y_1 + \lambda' \frac{dF}{dy} = 0,$$

$$R + \lambda z_1 + \lambda' \frac{dF}{dz} = 0.$$

Also $\qquad P = -x_2, \quad Q = -y_2, \quad R = -z_2;$

therefore

$$\left.\begin{aligned} \lambda x_1 + \lambda' \frac{dF}{dx} &= x_2 \\ \lambda y_1 + \lambda' \frac{dF}{dy} &= y_2 \\ \lambda z_1 + \lambda' \frac{dF}{dz} &= z_2 \end{aligned}\right\} \quad \text{............ (14),}$$

and we have the condition

$$\frac{dF}{dx}x_1 + \frac{dF}{dy}y_1 + \frac{dF}{dz}z_1 = 0 \quad \text{........ (15).}$$

Multiplying the equations (14) by x_1, y_1, z_1, respectively, we have

$$\lambda(x_1^2 + y_1^2 + z_1^2) = 0,$$

or

$$\lambda = 0,$$

and the equations (14) reduce themselves to

$$\frac{x_2}{\frac{dF}{dx}} = \frac{y_2}{\frac{dF}{dy}} = \frac{z_2}{\frac{dF}{dz}},$$

which are the equations to the shortest line required.

As a last example, I will take the case of a particle acted upon by forces $X\ Y\ Z$, and apply the principle of least action to find the differential equations of its path.

In this case

$$u = \int v\,ds,$$

where v is the velocity of the particle. Also

$$d.v^2 = 2(Xdx + Ydy + Zdz),$$

therefore

$$v^2 = 2\int(Xdx + Ydy + Zdz),$$

and

$$\frac{dv}{ds} = \frac{X}{v}x_1 + \frac{Y}{v}y_1 + \frac{Z}{v}z_1.$$

Also

$$P = \frac{X}{v} - vx_2,$$

$$Q = \frac{Y}{v} - vy_2,$$

$$R = \frac{Z}{v} - vz_2.$$

Therefore

$$\lambda x_1 + \frac{X}{v} - vx_2 = 0,$$

$$\lambda y_1 + \frac{Y}{v} - vy_2 = 0,$$

$$\lambda z_1 + \frac{Z}{v} - vz_2 = 0.$$

Multiplying these equations by x_1, y_1, z_1, respectively, and adding, we have

$$\lambda + \frac{dv}{ds} = 0,$$

therefore $$\frac{X}{v} = v\frac{d^2x}{ds^2} + \frac{du}{ds}\frac{dv}{ds} = \frac{d}{ds}\left(v\frac{dx}{ds}\right);$$

similarly $$\frac{Y}{v} = \frac{d}{ds}\left(v\frac{dy}{ds}\right),$$

$$\frac{Z}{v} = \frac{d}{ds}\left(v\frac{dz}{ds}\right),$$

which are the equations required, and are easily seen to be equivalent to only two.

I have already pointed out the manner in which the principles of this paper apply to double integrals, by observing that the unintegrable portion necessarily depends upon a variation in the direction of the normal to the surface in which the variables are supposed to be current coordinates; but I have not been able to extend my investigations to this case, so as to obtain any symmetrical formulæ.

I shall conclude this paper with a remark upon a point in the Theory of the Calculus of Variations, which has not, I think, been sufficiently explained.

The equation which has been denoted in this paper by $U = 0$ is, as is well known, not only that which belongs to the Calculus of Variations, but is given by Lagrange as the condition of Vdx being a perfect differential. There is however this difference between the two cases, that whereas in the first the equation $U = 0$ gives us a relation sought between the variables, in the latter it expresses a relation which will be satisfied identically if Vdx is a perfect differential; and I think it would be more proper to say that the equation expresses *a* condition of Vdx being a perfect differential rather than *the* condition, for it is nowhere proved that there may not be an indefinite number of other conditions. The principle adopted by Lagrange in his investigation is simply this, that if Vdx be a perfect differential, $\delta . Vdx$ will be so, a principle which may give us *a* condition, but cannot give us *the* only condition. Lagrange notices the fact of the equation $U = 0$ belonging to the two different cases, and says that the reason of the coincidence will be seen from the mode of investigation (*Leçons sur le Calcul des Fonctions*, Leç. xxi.); the way in which the coincidence does appear is, that the same method of investigation is applied to the two. This explanation, I confess, disappoints me, and I shall therefore endeavour to give one which will not only point out the fact but also the reason of it.

The condition that $u = \int V dx$ shall be a maximum or minimum is expressed by $\delta u = 0$, which equation signifies that the relation between x and y must be such that any slight change of that relation shall not alter the value of u; speaking geometrically, the value of u, as dependent upon the curve $U=0$, will not be changed if for $U=0$ we substitute any curve differing very slightly from $U = 0$. To fix our conceptions, suppose the limits fixed, then we may pass from one limit to the other, not only along the curve $U = 0$, but along any curve differing very slightly from it, and therefore along an indefinite number of curves. But again, the possibility of integrating the differential Vdx, without any given relation between x and y, merely expresses the fact that the value of the integral depends upon the limiting values of the variable, and is independent of the path by which we pass from one limit to the other. Hence in each case we have an integral which is unaltered in value when the curve connecting the limits is varied, and hence we can understand why the condition $U = 0$ belongs to each.

It might appear that there was this difference between the two cases, namely, that in the Calculus of Variations the equation $U = 0$ expresses that we may pass from one limit to the other along any curve *differing very slightly* from the required one, whereas if Vdx be a perfect differential the value of its integral is the same along *whatever* curve the integration is supposed to take place; but here comes in the remark which I have before made, that $U = 0$ is not *the* condition of Vdx being a perfect differential, but only *a* condition, which is investigated on the principle that Vdx will still continue to be a perfect differential if the relation between x and y is varied in an *indefinitely small* degree. Hence there is a perfect coincidence between the two cases, and therefore we have the same equation for each.

Cambridge, 1848.

SUGGESTION ON THE INTEGRATION OF RATIONAL FRACTIONS.

By Professor De Morgan.

The complexity of the operations by which

$$\int \phi x \, (x-a)^{-m} (x-b)^{-n} \ldots . dx$$

is found, ϕx being a rational function, is well known. A look

at the examples given for exercise by Peacock or Gregory will convince any one that such an instance as

$$\int (x^3 + 10x)(x-1)^{-7}(x-2)^{-5}\, dx$$

was considered by them as fairly beyond the working power even of the best students. And they were quite right, looking to the methods usually given for such a case.

The following method is easy for two inverse factors, and though the trouble increases rapidly with the number of factors, the *comparative* trouble, looking also at ordinary methods, really diminishes; as is almost always the case when a mechanical process is substituted for one which is also demonstrative of its result. It may be easily remembered, which is of importance in all rules of infrequent use. I need hardly say that had rational functions often required integration, it would long ago have been in all the books.

Throw the fraction $(x-a)^{-m}$ into the form $\{(b-a)+(x-b)\}^{-m}$. Use the binomial expansion up to the term in $(x-b)^{\cdot(n-1)}$, that is, as long as the multiplier $(x-b)^{-n}$ gives negative powers: and make this multiplication. The result is one part of $(x-a)^{-m}(x-b)^{-n}$. The other part is found by using $\{(a-b)+(x-a)\}^{-n}(x-a)^{-m}$ in the same manner.

For example, required $(x-3)^{-8}(x-7)^{-5}$; and for abbreviation, let $x-3$ and $x-7$ be represented by III and VII. Then we have, writing down only what is wanted,

$$(-4+\text{III})^{-5} = -4^{-5} - 5.4^{-6}.\text{III} - 15.4^{-7}\,\text{III}^2 - 35.4^{-8}.\text{III}^3 - 70.4^{-9}\text{III}^4$$
$$- 126.4^{-10}\,\text{III}^5 - 210.4^{-11}.\text{III}^6 - 330.4^{-12}\text{III}^7$$

$$(4+\text{VII})^{-8} = 4^{-8} - 8.4^{-9}\text{VII} + 36.4^{-10}\text{VII}^2 - 120.4^{-11}.\text{VII}^3 + 330.4^{-12}\text{VII}^4:$$

or

$$\frac{1}{(x-3)^8(x-7)^5} = -\frac{1}{4^5}\frac{1}{(x-3)^8} - \frac{5}{4^6}\frac{1}{(x-3)^7} - \frac{15}{4^7}\frac{1}{(x-3)^6}$$
$$-\frac{35}{4^8}\frac{1}{(x-3)^5} - \frac{70}{4^9}\frac{1}{(x-3)^4} - \frac{126}{4^{10}}\frac{1}{(x-3)^3} - \frac{210}{4^{11}}\frac{1}{(x-3)^2}$$
$$-\frac{330}{4^{12}}\frac{1}{x-3} + \frac{1}{4^8}\frac{1}{(x-7)^5} - \frac{8}{4^9}\frac{1}{(x-7)^4} + \frac{36}{4^{10}}\frac{1}{(x-7)^3}$$
$$-\frac{120}{4^{11}}\frac{1}{(x-7)^2} + \frac{330}{4^{12}}\frac{1}{x-7}.$$

When there is a function of x in the numerator, of a dimension less than $m+n$, it should be expanded in powers of

$x - a$ or $x - b$ by Horner's process. Thus for $\dfrac{x^3 - x}{(x-1)^2(x-2)^3}$ we should proceed as follows:

$$\begin{array}{llll} 1 & 0 & -1 & 0 \\ 1 & 2 & 3 & 6 \\ 1 & 4 & 11 & \\ 1 & 6 & & \end{array} \quad x^3 - x = (x-2)^3 + 6(x-2)^2 + 11(x-2) + 6$$

$$\frac{x^3 - x}{(x-1)^2(x-2)^3} = \frac{1}{(x-1)^2} + \frac{6}{(x-1)^2(x-2)} + \frac{11}{(x-1)^2(x-2)^2} + \frac{6}{(x-1)^2(x-2)^3},$$

each term of which must be separately dealt with. Or thus,

$$\begin{array}{llll} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & \\ 1 & 3 & & \end{array} \quad x^3 - x = (x-1)^3 + 3(x-1)^2 + 2(x-1)$$

$$\frac{x^3 - x}{(x-1)^2(x-2)^3} = \frac{x-1}{(x-2)^3} + \frac{3}{(x-2)^3} + \frac{2}{(x-1)(x-2)^3}$$

$$= \frac{1}{(x-2)^2} + \frac{4}{(x-2)^3} + \frac{2}{(x-1)(x-2)^3}.$$

I have purposely chosen a case in which there is a factor in both numerator and denominator, which is made visible in the second process, but not in the first.

If three or more factors be introduced, as in

$$(x-a)^{-m}(x-b)^{-n}(x-c)^{-p},$$

two of them may be first treated, and the third introduced into each term of the result, upon which the process may be repeated. This is more convenient than extending the law.

One proof of this rule is as follows. If we denote $(x-a)^{-m}$ $(x-b)^{-n}$ by $P_{m,n}$, the equation

$$(x-a)^{-1}(x-b)^{-1} = (b-a)^{-1}(x-b)^{-1} + (a-b)^{-1}(x-a)^{-1},$$

gives us

$$P_{m,n} = \alpha P_{m-1,n} + \beta P_{m,n-1}, \quad \alpha = (a-b)^{-1}, \quad \beta = (b-a)^{-1}.$$

This equation is well known, particularly in the theory of probabilities. The condition of our application of it is that, in forming our successive reductions, we must stop at $P_{0,n}$ and at $P_{m,0}$, whenever we obtain them. Accordingly we are not to allow $P_{0,n}$ to contribute anything to the formation of $P_{0,n-1}$.

Taking an instance, say $P_{4\cdot 3}$, we may represent the formation of the coefficients in the following manner:

	3	2	1	0
4	1	β	β^2	β^3
3	a	$2a\beta$	$3a\beta^2$	$3a\beta^3$
2	a^2	$3a^2\beta$	$6a^2\beta^2$	$6a^2\beta^3$
1	a^3	$4a^3\beta$	$10a^3\beta^2$	$10a^3\beta^3$
0	a^4	$4a^4\beta$	$10a^4\beta^2$	

The law of the numerical coefficients is, that each is the sum of the one above and of that on the left, except when one of the contributors is in a *zero* row or column, in which case its contribution is not made. It is now clear enough that

$$P_{m\cdot n} = a^m\left(P_{0,n} + m\beta P_{0,n-1} + m\,\frac{m+1}{2}\,\beta^2 P_{0,n-2} + \ldots\right) \text{as far as } P_{0,1}$$

$$+\,\beta^n\left(P_{m,0} + naP_{m-1,0} + n\,\frac{n+1}{2}\,a^2 P_{m-2,0} + \ldots\right) \text{as far as } P_{1,0},$$

from which the rule given may be easily deduced.

Though this proof may deserve attention, as being of a kind which sometimes succeeds when other methods fail, yet the proper connective proof is undoubtedly the easy one given in Lacroix (vol. II. p. 18), by which it is immediately shewn that $\phi x\,(x-a)^{-m}$ is of the form $A_0\,(x-a)^{-m} + \ldots + A_{m-1}(x-a)^{-1} + Q$, where $A_0 + A_1 h + \ldots + A_{m-1}h^{m-1}$ is a portion of the development of $\phi\,(a+h)$. Mr. Rawson, in a paper published in the *Mathematician* (vol. I. p. 265), has systematized the case in which ϕx is itself of the form $(x-b)^{-n}\,(x-c)^{-p}\ldots$, in a manner which should be examined by those who are desirous of cultivating this particular branch of integration. The direct method of proof did not recur to me when I wrote mine above, and I never saw Mr. Rawson's paper till I had finished what is here written. The use of the binomial

theorem will really, I believe, be found more expeditious and more safe than any method which does not refer to ready digested formulæ: successive reductions being made, commencing with two of the inverse factors.

I proceed to make a suggestion relative to the formation of the equations of reduction by which $\int(\phi x)^n x^m dx$ is usually found. These are generally given and used in a scattered matter: the student is left to invent each case by itself, and the processes generally involve much risk of error in the signs and constant divisors. All the reductions in the algebraical cases may be systematized by the use of the following formulæ, the application of which, from memory, will greatly reduce the risk of error.

Let $P = Ax^a + Bx^b + Cx^c + \ldots.$ $V_{m,n} = \int x^m P^n dx,$

$\alpha = m + 1 + na,\quad \beta = m + 1 + nb,\quad \gamma = m + 1 + nc,$ &c.

Then all the ordinary methods of reduction involve the use of

$$V_{m,n} = AV_{m+a,\,n-1} + BV_{m+b,\,n-1} + CV_{m+c,\,n-1} + \ldots.$$
$$x^{m+1}P^n = \alpha AV_{m+a,\,n-1} + \beta BV_{m+b,\,n-1} + \gamma CV_{m+c,\,n-1} + \ldots.$$

When P has only two terms, which is the most common case, the elimination of $V_{m+a,\,n-1}$ or $V_{m+b,\,n-1}$ is frequently used. This is best done, I think, in the particular case. Accordingly, we have four choices, each of which gives a reduction of m or of n, if not of both. These four connect

$(m, n)\,(m + a, n - 1,\ \ m + b, n - 1)$; $(m + a, n - 1)(m + b, n - 1)$;
$(m, n)(m + a, n - 1)$; (m, n) $(m + b, n - 1)$.

Of these, the first is comparatively useless; the second is always reductive of m (unless $a = b$, in which case there is no difficulty), and not of n; the third is always reductive of n, and, if $na\,(2m + a)$ be negative, of m also; the fourth is similarly related to b.

If m or n or both be fractional (say $m = m' + m_1$, $n = n' + n_1$, where m' and n' are integer, and m_1 and n_1 less than unity) much confusion and some trouble will be saved by adapting the forms of reduction to the rational part of the result. Let $W_{m,n} = x^{-m_1}P^{-n_1}V_{m,n}$, and we have the forms

$$W_{m,n} = AW_{m+a,\,n-1} + BW_{m+b,\,n-1};$$
$$x^{m'+1}P^{n'} = \alpha AW_{m+a,\,n-1} + \beta BW_{m+b,\,n-1};$$

by which $W_{m,n}$ may be reduced.

APPLICATION OF CERTAIN SYMBOLICAL REPRESENTATIONS OF FUNCTIONS TO INTEGRATION.

By the Rev. Brice Bronwin.

The present paper is a continuation of one by me in the 8th and 9th double Number of this *Journal*. Putting D for $\frac{d}{dr}$, and integrating from $r = -\infty$ to $r = r$, we have

$$D^{-1}\varepsilon^{rx} = \int dr\,\varepsilon^{rx} = x^{-1}\varepsilon^{rx},\quad D^{-2}\varepsilon^{rx} = x^{-2}\varepsilon^{rx},\ \&c.$$

Therefore

$$\Sigma a_n D^{-n}\varepsilon^{rx} = \Sigma a_n x^{-n}\varepsilon^{rx},\ \text{ or }\ \phi(x^{-1})\,\varepsilon^{rx} = \phi(D^{-1})\,\varepsilon^{rx}.$$

We may change r into $-r$, and integrate from $r = \infty$ to $r = r$. Thus, suitably changing the function ϕ, we have

$$\left.\begin{aligned}\phi(x)\,\varepsilon^{rx} &= \phi(D)\,\varepsilon^{rx}\\ \phi(x)\,\varepsilon^{-rx} &= \phi(-D)\,\varepsilon^{-rx}\end{aligned}\right\}\ \ldots\ldots\ldots\ldots (1).$$

To give an example of the application of the second of these, let

$$\phi(x) = \frac{1}{1+x^2}.$$

Then $$y = \int_0^\infty \frac{dx\,\varepsilon^{-rx}\cos nx}{1+x^2} = (D^2+1)^{-1}\int_0^\infty dx\,\varepsilon^{-rx}\cos nx$$

$$= (D^2+1)^{-1}\frac{r}{n^2+r^2} = \frac{1}{2\sqrt{(-1)}}\left\{\frac{1}{D-\sqrt{(-1)}} - \frac{1}{D+\sqrt{(-1)}}\right\}\frac{r}{n^2+r^2}$$

$$= \frac{1}{2\sqrt{(-1)}}\,\varepsilon^{r\sqrt{(-1)}}D^{-1}\left(\frac{r\varepsilon^{-r\sqrt{(-1)}}}{n^2+r^2}\right) - \frac{1}{2\sqrt{(-1)}}\,\varepsilon^{-r\sqrt{(-1)}}D^{-1}\left(\frac{r\varepsilon^{r\sqrt{(-1)}}}{n^2+r^2}\right).$$

The reduction is here made by the formula

$$f(D+a)\,\phi(r) = \varepsilon^{-ar}f(D)\,\varepsilon^{ar}\,\phi(r).$$

Make $r = 0$, and the above becomes

$$y = \int_0^\infty \frac{dx\cos nx}{1+x^2} = -\int_\infty^0 \frac{rdr\sin r}{n^2+r^2} = \int_0^\infty \frac{rdr\sin r}{n^2+r^2}$$

$$= (\text{by changing } r \text{ into } nx)\int_0^\infty \frac{xdx\sin nx}{1+x^2} = -\frac{dy}{dn}.$$

Therefore $$\frac{dy}{dn} + y = 0,\ \text{ and }\ y = A\varepsilon^{-n} = \frac{\pi}{2}\varepsilon^{-n}.$$

If we take for $\phi(x)$ such a function that $\phi(D)$, or $\phi(-D)$ will reduce to a number of terms of the form $\frac{a}{D+b}$; we can always either integrate or transform the integral.

From the second of (1), we have

$$\int_0^\infty x^{n-1}dx\,\phi(x)\,\varepsilon^{-rx} = \phi(-D)\int_0^\infty x^{n-1}dx\,\varepsilon^{-rx} = \phi(-D)\,\frac{\Gamma(n)}{r^n};$$

also $\left(\int_\infty^0 dx\right)^n \phi(x)\,\varepsilon^{-rx} = \phi(-D)\left(\int_\infty^0 dx\right)^n \varepsilon^{-rx} = \phi(-D)\,\frac{(-1)^n}{r^n}$.

Therefore $\int_0^\infty x^{n-1}dx\,\phi(x)\,\varepsilon^{-rx} = \frac{\Gamma(n)}{(-1)^n}\left(\int_\infty^0 dx\right)^n \phi(x)\,\varepsilon^{-rx}$.

Making $r = 0$, this becomes the known formula

$$\int_0^\infty x^{n-1}dx\,\phi(x) = \frac{\Gamma(n)}{(-1)^n}\left(\int_\infty^0 dx\right)^n \phi(x).$$

And, making $r = 0$ in the second member, after the operations on it are performed, we have also this singular theorem,

$$\int_0^\infty x^{n-1}dx\,\phi(x) = \Gamma(n)\,\phi\left(-\frac{d}{dr}\right)\frac{1}{r^n} \;\ldots\ldots\; (2).$$

We may employ (1) to integrate by series. Thus

$$\int_0^\infty dx\,\phi(x)\,\varepsilon^{-rx} = \phi(-D)\int_0^\infty dx\,\varepsilon^{-rx} = \phi(-D)\,\frac{1}{r}.$$

Make $\phi(x) = \frac{1}{1+x}$, then

$$\int_0^\infty \frac{dx\,\varepsilon^{-rx}}{1+x} = (1-D)^{-1}\frac{1}{r} = -\,\varepsilon^r\int_\infty^r \frac{dr}{r}\,\varepsilon^{-r} = \varepsilon^r\int_r^\infty \frac{dr}{r}\,\varepsilon^{-r}$$

$$= \varepsilon^r\int_0^\infty \frac{dr}{r}\,\varepsilon^{-r} - \varepsilon^r\int_0^r \frac{dr}{r}\,\varepsilon^{-r}.$$

Developing ε^{-r}, integrating, and making $C = \int_0^\infty \frac{dr}{r}\,\varepsilon^{-r} - \log 0$, a finite constant, we find

$$\int_0^\infty \frac{dx\,\varepsilon^{-rx}}{1+x} = \varepsilon^r\left(C - \log r + r - \frac{r^2}{2^2} + \frac{r^3}{2.3^2} - \ldots\right).$$

In (1) we may change x into a function of x. Change it into x^2, then $\phi(x^2)\,\varepsilon^{-rx^2} = \phi(-D)\,\varepsilon^{-rx^2}$; and we find

$$\int_0^\infty \frac{dx\,\varepsilon^{-rx^2}}{1+x^2} = (1-D)^{-1}\int_0^\infty dx\,\varepsilon^{-rx^2} = -\,\frac{\pi^{\frac{1}{2}}}{2}(D-1)^{-1}\frac{1}{r^{\frac{1}{2}}}$$

$$= -\,\frac{\pi^{\frac{1}{2}}}{2}\,\varepsilon^r\int_\infty^r \frac{dr}{r^{\frac{1}{2}}}\,\varepsilon^{-r} = \pi^{\frac{1}{2}}\,\varepsilon^r\int_{r^{\frac{1}{2}}}^\infty dx\,\varepsilon^{-x^2}.$$

The last step is made by changing r into x^2.

Multiply the first of (1) by $f\left(\frac{d}{dx}\right)$, and change D into $\frac{d}{dr}$; then

$$f\left(\frac{d}{dx}\right)\phi(x)\,\varepsilon^{rx} = \phi\left(\frac{d}{dr}\right)f\left(\frac{d}{dx}\right)\varepsilon^{rx}.$$

But $f\left(\frac{d}{dx}\right)\varepsilon^{rx} = f(r)\,\varepsilon^{rx}$ by the same.

Therefore
$$\left.\begin{aligned} f\left(\frac{d}{dx}\right)\phi(x)\,\varepsilon^{rx} &= \phi\left(\frac{d}{dr}\right)f(r)\,\varepsilon^{rx} \\ f\left(\frac{d}{dx}\right)\phi(x)\,\varepsilon^{-rx} &= \phi\left(-\frac{d}{dr}\right)f(-r)\,\varepsilon^{rx} \end{aligned}\right\} \ldots\ldots (3),$$

the second being formed from the first by changing r into $-r$. If $f\left(\frac{d}{dx}\right)$ contain negative powers of $\frac{d}{dx}$, the first of these must be integrated from $x=-\infty$ to $x=x$, and the second from $x=\infty$ to $x=x$. These are beautiful theorems: I cannot stop to make any particular application of them; but we easily find from them the following singular formulæ:

$$\int_0^\infty dx \sin mx\, f\left(\frac{d}{dx}\right)\phi(x)\,\varepsilon^{-rx} = \phi\left(-\frac{d}{dr}\right)\frac{mf(-r)}{m^2-r^2},$$

$$\int_0^\infty dx \cos mx\, f\left(\frac{d}{dx}\right)\phi(x)\,\varepsilon^{-rx} = \phi\left(-\frac{d}{dr}\right)\frac{rf(-r)}{m^2+r^2}.$$

In the first of (1) change $\phi(x)$ into $\phi(\varepsilon^x)$, then ε^x into x; and because $\varepsilon^D = 1+\Delta = \mathrm{E}$; there results

$$\phi(x)\,x^r = \phi(\mathrm{E})\,x^r \ldots\ldots \ldots\ldots (4).$$

From this we obtain

$$\int_0^1 x^{r-1}dx\,(1-x)^{n-1}\phi(x) = \phi(\mathrm{E})\int_0^1 x^{r-1}dx(1-x)^{n-1} = \phi(\mathrm{E})\,\frac{\Gamma(r)\,\Gamma(n)}{\Gamma(r+n)}.$$

As a particular example,

$$\int_0^1 \frac{x^{r-1}dx\,(1-x)^{n-1}}{(a+x)^{r+n}} = \frac{1}{(a+\mathrm{E})^{r+n}}\,\frac{\Gamma(r)\,\Gamma(n)}{\Gamma(r+n)}$$

$$= a^{-(r+n)}\left\{1-(r+n)\,a^{-1}\mathrm{E} + \frac{(r+n)(r+n+1)}{2}\,a^{-2}\mathrm{E}^2 - \ldots\right\}\frac{\Gamma(r)\,\Gamma(n)}{\Gamma(r+n)}$$

$$= a^{-(r+n)}\left\{1 - ra^{-1} + \frac{r(r+1)}{2}\,a^{-2} - \ldots\right\}\frac{\Gamma(r)\,\Gamma(n)}{\Gamma(r+n)}$$

$$= \frac{1}{a^n(1+a)^r}\,\frac{\Gamma(r)\,\Gamma(n)}{\Gamma(r+n)}.$$

The reduction is here made thus:

$$\mathrm{E}\Gamma(r) = r\Gamma(r),\quad \mathrm{E}^2\Gamma(r) = r(r+1)\Gamma(r),\ \&c.$$

Other similar integrals may be obtained with equal facility. The integration, being prior to the development, greatly facilitates the attainment of the result. The next example leads to a useful theorem.

$$\int_0^1 x^{r-1}dx(1-x)^{n-1}\phi(x) = \phi(\mathrm{E})\frac{\Gamma(r)\Gamma(n)}{\Gamma(r+n)} = \phi(\mathrm{E})\frac{\Gamma(n)}{r(r+1)\ldots(r+n-1)}.$$

But $\int_0^x x^{r-1}dx\,\phi(x) = \phi(\mathrm{E})\frac{x^r}{r}$, $\iint x^{r-1}dx^2\phi(x) = \phi(\mathrm{E})\frac{x^{r+1}}{r(r+1)}$, &c.;

and therefore

$$\int_0^1 x^{r-1}dx(1-x)^{n-1}\phi(x) = \Gamma(n)\left(\int_0^1 dx\right)^n\{x^{r-1}\phi(x)\}.$$

Or, if $r=1$,

$$\int_0^1 dx(1-x)^{n-1}\phi(x) = \Gamma(n)\left(\int_0^1 dx\right)^n\phi(x)\ldots\ldots\ldots(5).$$

In (4) we may change x into ε^x, and ε^{-x}; when it gives

$$\phi(\varepsilon^x)\,\varepsilon^{rx} = \phi(\mathrm{E})\,\varepsilon^{rx},\quad \phi(\varepsilon^{-x})\,\varepsilon^{-rx} = \phi(\mathrm{E})\,\varepsilon^{-rx}.$$

From these we deduce, exactly as (3) were obtained,

$$\left.\begin{aligned} f\left(\frac{d}{dx}\right)\phi(\varepsilon^x)\,\varepsilon^{rx} &= \phi(\mathrm{E})f(r)\,\varepsilon^{rx} \\ f\left(\frac{d}{dx}\right)\phi(\varepsilon^{-x})\,\varepsilon^{-rx} &= \phi(\mathrm{E})f(-r)\,\varepsilon^{-rx} \end{aligned}\right\}\ldots\ldots(6).$$

In the first of (1), change x into $\log x = lx$, and it becomes

$$x^r\phi(lx) = \phi(D)\,x^r.$$

From this, supposing r greater than nothing and less than unity, we find

$$\int_0^\infty \frac{x^{r-1}dx\,\phi(lx)}{1+x} = \phi(D)\int_0^\infty\frac{x^{r-1}dx}{1+x} = \phi(D)\frac{\pi}{\sin r\pi}$$

$$\int_0^\infty \frac{x^{r-1}dx\,\phi(lx)}{1-x} = \phi(D)\int_0^\infty\frac{x^{r-1}dx}{1-x} = \phi(D)\frac{\pi}{\tan r\pi}.$$

But we have still to perform the operations denoted by $\phi(D)$ on the second members. This may be done when ϕ denotes an integer power, especially if positive.

We may change the first of (1) into $\phi(x^2)\,\epsilon^{rx} = \phi(D^2)\,\epsilon^{rx}$; and then x into $x\sqrt{(-1)}$ and $-x\sqrt{(-1)}$, and take half the sum and difference of the results. We thus find

$$\phi(-x^2)\cos rx = \phi(D^2)\cos rx, \quad \phi(-x^2)\sin rx = \phi(D^2)\sin rx,$$

which may be changed into

$$\phi(x^2)\cos rx = \phi(-D^2)\cos rx, \quad \phi(x^2)\sin rx = \phi(-D^2)\sin rx.$$

And these will give, as before,

$$\left.\begin{aligned} f\left(\frac{d^2}{dx^2}\right)\phi(x^2)\sin rx &= \phi\left(-\frac{d^2}{dr^2}\right)f(-r^2)\sin rx \\ f\left(\frac{d^2}{dx^2}\right)\phi(x^2)\cos rx &= \phi\left(-\frac{d^2}{dr^2}\right)f(-r^2)\cos rx \end{aligned}\right\}\ldots(7).$$

As $\sin rx$ and $\cos rx$ become zero when $r = \infty$, or $x = \infty$; we may suppose this the first limit of integration relative to these quantities.

We may introduce more symbols of operation. For since

$$\phi(x)\,\epsilon^{-rx} = \phi(D)\,\epsilon^{-rx}, \quad f(x^2)\sin nx = f(-D'^2)\sin nx,$$
$$\text{and } f(x^2)\cos nx = f(-D'^2)\cos nx;$$

D operating on r, and D' on n, we shall have

$$\phi(x)f(x^2)\,\epsilon^{-rx}\sin nx = \phi(-D)f(-D'^2)\,\epsilon^{-rx}\sin nx,$$
$$\phi(x)f(x^2)\,\epsilon^{-rx}\cos nx = \phi(-D)f(-D'^2)\,\epsilon^{-rx}\cos nx;$$

and therefore we shall have likewise

$$\int_0^\infty dx\,\phi(x)f(x^2)\,\epsilon^{-rx}\sin nx = \phi(-D)f(-D'^2)\frac{n}{n^2+r^2},$$

$$\int_0^\infty dx\,\phi(x)f(x^2)\,\epsilon^{-rx}\cos nx = \phi(-D)f(-D'^2)\frac{r}{n^2+r^2}.$$

If $\phi(x)$ and $f(x^2)$ be such that $\phi(-D)$ and $f(-D'^2)$ can be resolved into terms of the form $(D+a)^{-1}$, $(D'+a')^{-1}$; the integrations of the second members will be effected as before explained, or they will be transformed into different integrals from those contained in the first members.

By combining

$$\phi(x)\,x^r = \phi(\mathrm{E})\,x^r, \quad f(x^2)\sin nx = f(-D^2)\sin nx,$$
$$\text{and } f(x^2)\cos nx = f(-D^2)\cos nx,$$

where E operates on r, D on n; we have

$$x^r\phi(x)f(x^2)\sin nx = \phi(E)f(-D^2)\,x^r\sin nx,$$
$$x^r\phi(x)f(x^2)\cos nx = \phi(E)f(-D^2)\,x^r\cos nx.$$

These perhaps would not be so easy of application to integration as the last. Those which follow next will be very suitable for the purpose. By combining $\phi(x)\, x^r = \phi(E)\, x^r$, $f(x)\, \epsilon^{nx} = f(D)\, \epsilon^{nx}$, and $f(x)\, \epsilon^{-nx} = f(-D)\, \epsilon^{-nx}$, E operating on r, and D on n; we have

$$x^r\phi(x) f(x)\, \epsilon^{nx} = \phi(E) f(D)\, x^r \epsilon^{nx},$$

$$x^r\phi(x) f(x)\, \epsilon^{-nx} = \phi(E) f(-D)\, x^r \epsilon^{-nx}.$$

The second of these gives us

$$\int_0^\infty x^{r-1}dx\, \phi(x) f(x)\, \epsilon^{-nx} = \phi(E) f(-D) \int_0^\infty x^{r-1}dx\, \epsilon^{-nx}$$

$$= \phi(E) f(-D) \frac{\Gamma(r)}{n^r}.$$

As particular cases, make $f(x) = \dfrac{1}{1+x}$ and $\dfrac{1}{1+x^2}$. In these cases

$$f(-D) = (1-D)^{-1} = -\epsilon^n \int dn\, \epsilon^{-n},$$

and $f(-D) = (1+D^2)^{-1} = \dfrac{1}{2\sqrt{(-1)}}\,(D-\sqrt{(-1)})^{-1} - \dfrac{1}{2\sqrt{(-1)}}\,(D+\sqrt{(-1)})^{-1}$

$$= \frac{1}{2\sqrt{(-1)}}\, \epsilon^{n\sqrt{(-1)}} \int dn\, \epsilon^{-n\sqrt{(-1)}} - \frac{1}{2\sqrt{(-1)}}\, \epsilon^{-n\sqrt{(-1)}} \int dn\, \epsilon^{n\sqrt{(-1)}}$$

$$= \sin n \int dn \cos n - \cos n \int dn \sin n.$$

Therefore $\displaystyle\int_0^\infty \frac{x^{r-1}dx\, \phi(x)\, \epsilon^{-nx}}{1+x} = -\phi(E)\, \Gamma(r)\, \epsilon^n \int_\infty^n \frac{dn}{n^r}\, \epsilon^{-n}$

$$= \phi(E)\, \Gamma(r)\, \epsilon^n \int_n^\infty \frac{dn}{n^r}\, \epsilon^{-n}.$$

$$\int_0^\infty \frac{x^{r-1}dx\, \phi(x)\, \epsilon^{-nx}}{1+x^2} = \phi(E)\Gamma(r)\left\{\sin n \int_\infty^n \frac{dn}{n^r} \cos n - \cos n \int_\infty^n \frac{dn}{n^r} \sin n\right\}$$

$$= \phi(E)\, \Gamma(r) \left\{\cos n \int_n^\infty \frac{dn}{n^r} \sin n - \sin n \int_n^\infty \frac{dn}{n^r} \cos n\right\},$$

where we have still to operate with E on the second members. If we make $n = 0$, they become

$$\int_0^\infty \frac{x^{r-1}dx\, \phi(x)}{1+x} = \phi(E)\, \Gamma(r) \int_0^\infty \frac{dn}{n^r}\, \epsilon^{-n} = \phi(E)\, \Gamma(r)\, \Gamma(1-r)$$

$$= \phi(E) \frac{\pi}{\sin r\pi}. \quad \int_0^\infty \frac{x^{r-1}dx\, \phi(x)}{1+x^2} = \phi(E)\, \Gamma(r) \int_0^\infty \frac{dn}{n^r} \sin n$$

$$= \phi(E)\, \Gamma(r)\, \Gamma(1-r) \sin(1-r)\tfrac{1}{2}\pi = \phi(E)\, \frac{\pi}{2 \sin \frac{1}{2} r\pi}.$$

When ϕ denotes a power, whether integer or fractional; the operations $\phi(E)$ can be performed on the second member.

I shall now give some formulæ which, though not closely connected with what precedes, appear deserving of notice.

Let $\qquad \phi = ak^x + a_1k^{x+h} + a_2k^{x+2h} + \dots.$

Putting D for $\dfrac{d}{dk}$, and performing successively the operation (kD);

$$(kD)\,\phi = ak^x x + a_1k^{x+h}(x+h) + a_2k^{x+2h}(x+2h) + \dots.$$

$$(kD)^2\,\phi = ak^x x^2 + a_1k^{x+h}(x+h)^2 + a_2k^{x+2h}(x+2h)^2 + \dots.$$

$$(kD)^3\,\phi = ak^x x^3 + a_1k^{x+h}(x+h)^3 + a_2k^{x+2h}(x+2h)^3 + \dots.$$

&c.

Therefore

$$f(kD)\,\phi = ak^x f(x) + a_1k^{x+h}f(x+h) + a_2k^{x+2h}f(x+2h) + \dots.$$

The following are particular cases of this:

$$\phi = ak^x + a_1k^{x+1} + \dots \quad f(kD)\,\phi = ak^x f(x) + a_1k^{x+1}f(x+1) + \dots$$

$$\phi = a + a_1k + a_2k^2 + \dots \quad f(kD)\,\phi = af(0) + a_1kf(1) + a_2k^2f(2) + \dots$$

$$\phi = \varepsilon^k = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \dots f(kD)\,\varepsilon^k = f(0) + \frac{k}{1}f(1) + \frac{k^2}{1.2}f(2) + \dots$$

The last may be written

$$f(kD)\,\varepsilon^k = \varepsilon^{kE}f(0) \dots\dots\dots\dots\dots (8),$$

where E operates on 0. And this is equivalent to

$$f(0) + \frac{k}{1}f(1) + \frac{k^2}{1.2}f(2) + \dots.$$

$$= \varepsilon^k\left\{f(0) + \frac{k}{1}\Delta f(0) + \frac{k^2}{1.2}\Delta^2 f(0) + \dots.\right\},$$

as will appear by putting $1 + \Delta$ for E, and developing.

In the general case, make $\psi(k) = a + a_1k^h + a_2k^{2h} + \dots.$ and $\phi = k^x\psi(k)$, and let E operate upon x; then the second member becomes

$$(a + a_1E^h + a_2E^{2h} + \dots.)\,\{k^x f(x)\} = \psi(E)\,\{k^x f(x)\}.$$

Therefore the formula itself is

$$f\left(k\frac{d}{dk}\right)\{k^x\psi(k)\} = \psi(E)\,\{k^x f(x)\} \dots\dots (9).$$

Resuming $\phi = ak^x + a_1k^{x+h} + a_2k^{x+2h} + \ldots$, and differentiating successively,

$$D\phi = axk^{x-1} + a_1(x+h)\,k^{x+h-1} + a_2(x+2h)\,k^{x+2h-1}\ldots.$$

$$D^2\phi = ax(x-1)\,k^{x-2} + a_1(x+h)(x+h-1)\,k^{x+h-2}$$
$$+ a_2(x+2h)(x+2h-1)\,k^{x+2h-2} + \ldots.$$

$$D^3\phi = ax(x-1)(x-2)k^{x-3}\,a_1(x+h)(x+h-1)(x+h-2)k^{x+h-3} + \ldots.$$

&c.

Multiply these respectively by 0^n, $\frac{k}{1}\Delta 0^n$, $\frac{k^2}{1.2}\Delta^2 0^n$, &c., and sum vertically, taking the sums of the coefficients of the second members by the theorem

$$x^n = 0^n + \frac{x}{1}\Delta 0^n + \frac{x(x-1)}{1.2}\Delta^2 0^n + \ldots.;$$

we have

$$\phi 0^n + \frac{k}{1}D\phi\Delta 0^n + \frac{k^2}{1.2}D^2\phi\Delta^2 0^n + \ldots. + \frac{k^n}{1.2\ldots n}D^n\phi\Delta^n 0^n$$
$$= ak^x x^n + a_1k^{x+h}(x+h)^n + a_2k^{x+2h}(x+2h)^n + \ldots.$$

Thus the sum of the series constituting the second member is given in finite terms when ϕ is a finite function.

But by (e) of my former paper, $D^m\phi(k) = \phi(k + D')\,0'^m$, where D' stands for $\frac{d}{d0'}$. Therefore the first member of the above becomes

$$\phi(k+D')\left\{0^n + \frac{k0'}{1}\Delta 0^n + \frac{k^2 0'^2}{1.2}\Delta^2 0^n + \ldots. + \frac{k^n 0'^n}{1.2\ldots n}\Delta^n 0^n\right\}$$
$$= \phi(k + D')\,\varepsilon^{k0'\Delta}0^n.$$

And the second member is equivalent to $(kD)^n\,\phi(k)$. Therefore

$$(kD)^n\,\phi(k) = \phi(k + D')\,\varepsilon^{k0'\Delta}\,0^n \;\ldots\ldots\; (10).$$

As a particular case, let $\phi(k) = k^x\varepsilon^k$, and the last becomes

$$(kD)^n\,(k^x\varepsilon^k) = (k + D')^x\,\varepsilon^{k+D'}\,(\varepsilon^{k0'\Delta})\,0^n.$$

Making $r = 0$, the first of (1) becomes $\phi(x) = \phi(D)\,\varepsilon^{0x}$. Therefore $\Delta^n\phi(x) = \phi(D)\,\varepsilon^{0x}(\varepsilon^0 - 1)^n$

$$= \phi(D)\,\varepsilon^{0x}\left\{\varepsilon^{n0} - \frac{n}{1}\varepsilon^{(n-1)0} + \frac{n(n-1)}{1.2}\varepsilon^{(n-2)0} - \ldots.\right\}.$$

Expanding ε^{n0}, $\varepsilon^{(n-1)0}$, &c., we find the coefficients of 0, 0^2, 0^3, &c. to be

$$\frac{\Delta'^n 0'}{1},\quad \frac{\Delta'^n 0'^2}{1.2},\quad \frac{\Delta'^n 0'^3}{1.2.3},\ \text{\&c.},$$

Δ' operating on $0'$, as D on 0. Therefore, omitting the terms that vanish,

$$\Delta^n\phi(x) = \phi(D)\,\varepsilon^{0x}\left\{\frac{0^n}{\Gamma(n+1)}\,\Delta'^n 0'^n + \frac{0^{n+1}}{\Gamma(n+2)}\,\Delta'^n 0'^{n+1} + \ldots\right\}.$$

But $$\phi(D)\,\varepsilon^{0x}0^n = \frac{d^n\phi(x)}{dx^n} = \phi(x+D)\,0^n.$$

Therefore

$$\Delta^n\phi(x) = \phi(x+D)\,\Delta'^n\left\{\frac{0^n 0'^n}{\Gamma(n+1)} + \frac{0^{n+1}0'^{n+1}}{\Gamma(n+2)} + \frac{0^{n+2}0'^{n+2}}{\Gamma(n+3)} + \ldots\right\} \ldots (11).$$

In coming to a conclusion, I would observe that the theorems (1) and (4) may be made much more general; so that the functional symbol ϕ may be much more general. Thus if ρ denotes such an operation on r that $\rho^k\varepsilon^{rx} = x^k\varepsilon^{rx}$, and θ such that $\theta^k\varepsilon^{-rx} = x^{k-rx}$; then

$$\phi(x)\,\varepsilon^{rx} = \phi(\rho)\,\varepsilon^{rx},\quad \phi(x)\,\varepsilon^{-rx} = \phi(\theta)\,\varepsilon^{-rx} \ldots (12),$$

k being any quantity whatever.

And, in like manner, if β denote such an operation on r that $\beta^k x^r = x^k x^r$, then

$$\phi(x)\,x^r = \phi(\beta)\,x^r \ldots\ldots\ldots\ldots (13),$$

k being here as general as in the last case.

Since writing the above, the following handsome theorem has occurred to me. Multiply the first of (3) by ε^{r-x}, which gives

$$\varepsilon^{-rx} f\left(\frac{d}{dx}\right)\varepsilon^{rx}\phi(x) = \varepsilon^{-rx}\phi\left(\frac{d}{dr}\right)\varepsilon^{rx}f(r).$$

And this, by a well known theorem, is equivalent to

$$f\left(\frac{d}{dx}+r\right)\phi(x) = \phi\left(\frac{d}{dr}+x\right)f(r) \ldots (14),$$

the integrations, if any, from $x = -\infty$ to $x = x$, and from $r = -\infty$ to $r = r$.

January 3, 1848.

ON THE STRENGTH OF MATERIALS, AS INFLUENCED BY THE EXISTENCE OR NONEXISTENCE OF CERTAIN MUTUAL STRAINS AMONG THE PARTICLES COMPOSING THEM.

By JAMES THOMSON, Jun., M.A., College, Glasgow.

MY principal object in the following paper is to shew that the absolute strength of any material composed of a substance possessing ductility (and few substances, if any, are entirely devoid of this property), may vary to a great extent, according to the state of tension or relaxation in which the particles have been made to exist when the material as a whole is subject to no external strain.

Let, for instance, a cylindrical bar of malleable iron, or a piece of iron wire, be made red hot, and then be allowed to cool. Its particles may now be regarded as being all completely relaxed. Let next the one end of the bar be fixed, and the other be made to revolve by tortion, till the particles at the circumference of the bar are strained to the utmost extent of which they can admit without undergoing a permanent alteration in their mutual connexion.* In this condition, equal elements of the cross section of the bar afford resistances proportional to the distances of the elements from the centre of the bar; since the particles are displaced from their positions of relaxation through spaces which are proportional to the distances of the particles from the centre. The couple which the bar now resists, and which is equal to the sum of the couples due to the resistances of all the elements of the section, is that which is commonly assumed as the measure of the strength of the bar. For future reference, this couple may be denoted by L, and the angle through which it has twisted the loose end of the bar by Θ.

The twisting of the bar may, however, be carried still farther, and during the progress of this process the outer particles will yield in virtue of their ductility, those towards the interior assuming successively the condition of greatest tension; until, when the twisting has been sufficiently continued, all the particles in the section, except those quite close to the centre, will have been brought to afford their utmost resistance. Hence, if we suppose that no change in

* I here assume the existence of a definite "*elastic limit*," or a limit within which if two particles of a substance be displaced, they will return to their original relative positions when the disturbing force is removed. The opposite conclusion, to which Mr. Hodgkinson seems to have been led by some interesting experimental results, will be considered at a more advanced part of this paper.

the hardness of the substance composing the material has resulted from the sliding of its particles past one another; and that, therefore, all small elements of the section of the bar afford the same resistance, no matter what their distances from the centre may be, it is easy to prove that the total resistance of the bar is now $\frac{4}{3}$ of what it was in the former case; or, according to the notation already adopted, it is now $\frac{4}{3}L$.*

If, after this, all external strain be removed from the bar, it will assume a position of equilibrium, in which the outer particles will be strained in the direction opposite to that in which it was twisted and the inner ones in the same direction as that of the twisting, the two sets of opposite couples thus produced among the particles of the bar balancing one another. It is easy to shew that the line of separation between the particles strained in the one direction, and those

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* To prove this, let r be the radius of the bar, η the utmost force of a unit of area of the section to resist a strain tending to make the particles slide past one another; or to resist a shearing strain, as it is commonly called. Also let the section of the bar be supposed to be divided into an infinite number of concentric annular elements; the radius of any one of these being denoted by x, and its area by $2\pi x dx$.

Now, when only the particles at the circumference are strained to the utmost; and when, therefore, the forces on equal areas of the various elements are proportional to the distances of the elements from the centre, we have $\eta \frac{x}{r}$ for the force of a unit of area at the distance x from the centre. Hence the total tangential force of the element is

$$= 2\pi x dx . \eta \frac{x}{r},$$

and the couple due to the same element is

$$= x . 2\pi x dx . \eta \frac{x}{r} = 2\pi\eta \frac{1}{r} . x^3 dx:$$

and therefore the total couple, which has been denoted above by L, is

$$= 2\pi\eta \frac{1}{r} \int_0^r x^3 dx,$$

that is $$L = \tfrac{1}{2}\pi\eta r^3 \ldots\ldots\ldots\ldots (a).$$

Next, when the bar has been twisted so much that all the particles in its section afford their utmost resistance, we have the total tangential force of the element $= 2\pi x dx . \eta$, and the couple due to the same element

$$= x . 2\pi x dx . \eta = 2\pi\eta . x^2 dx.$$

Hence the total couple due to the entire section is

$$= 2\pi\eta \int_0^r x^2 dx = \tfrac{2}{3}\pi\eta r^3.$$

But this quantity is $\frac{4}{3}$ of the value of L in formula (a). That is, the couple which the bar resists in this case is $\frac{4}{3}L$, or $\frac{4}{3}$ of that which it resisted in the former case.

in the other, is a circle whose radius is $\frac{3}{4}$ of the radius of the bar. The particles in this line are evidently subject to no strain* when no external couple is applied. The bar with its new molecular arrangement may now be subjected, *as often as we please*,† to the couple $\frac{4}{3}L$, without undergoing any farther alteration; and therefore its ultimate strength to resist torsion, in the *direction of the couple L*, has been considerably increased. Its strength to resist torsion in the opposite direction has, however, by the same process, been much diminished: for, as soon as its free extremity has been made to revolve backwards through an angle of $\frac{2}{3}\Theta$ from the position of equilibrium, the particles at the circumference will have suffered the utmost displacement of which they can admit without undergoing permanent alteration. Now it is easy to prove that the couple required to produce a certain angle of torsion is the same in the new state of the bar as in the old.‡ Hence the ultimate strength of the bar when twisted backwards, is represented by a couple amounting to only $\frac{2}{3}L$. But, as we have seen, it is $\frac{4}{3}L$ when the wire is twisted forwards. That is, then, *The wire in its new state has twice as much strength to resist torsion in the one direction as it has to resist it in the other.*

Principles quite similar to the foregoing, operate in regard to beams subjected to cross strain. As, however, my chief object at present is to point out the existence of such principles, to indicate the mode in which they are to be applied, and to shew their great practical importance in the determination of the strength of materials, I need not enter fully

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* Or at least they are subject to *no strain of torsion* either in the one direction or in the other; though they may perhaps be subject to a strain of compression or extension in the direction of the length of the bar. This, however, does not fall to be considered in the present investigation.

† This statement, if not strictly, is at least extremely nearly true: since from the experiments made by Mr. Fairbairn and Mr Hodgkinson on cast iron (See various *Reports of the British Association*), we may conclude that the metals are influenced only in an extremely slight degree by time. Were the bars composed of some substance, such as sealing wax, or hard pitch, possessing a sensible amount of viscidity, the statement in the text would not hold good.

‡ To prove this, let the bar be supposed to be divided into an infinite number of elementary concentric tubes (like the so called annual rings of growth in trees). To twist each of these tubes through a certain angle, the same couple will be required whether the tube is already subject to the action of a couple of any moderate amount in either direction, or not. Hence, to twist them all, or what is the same thing, to twist the whole bar, through a certain angle, the same couple will be required whether the various elementary tubes be or be not relaxed, when the bar as a whole is free from external strain.

into their application in the case of cross strain. The investigation in this case closely resembles that in the case of torsion, but is more complicated on account of the different ultimate resistances afforded by any material to tension and to compression, and on account of the numerous varieties in the form of section of beams which for different purposes it is found advisable to adopt. I shall therefore merely make a few remarks on this subject.

If a bent bar of wrought iron, or other ductile material, be straightened, its particles will thus be put into such a state, that its strength to resist cross strain, in the direction towards which it has been straightened, will be very much greater than its strength to resist it in the opposite direction, each of these two resistances being entirely different from that which the same bar would afford, were its particles all relaxed when the entire bar is free from external strain. The actual ratios of these various resistances depend on the comparative ultimate resistances afforded by the substance to compression and extension; and also, in a very material degree, on the form of the section of the bar. I may however state that in general the variations in the strength of a bar to resist cross strain, which are occasioned by variations in its molecular arrangement, are much greater even than those which have already been pointed out as occurring in the strength of bars subjected to torsion.

What has been already stated is quite sufficient to account for many very discordant and perplexing results which have been arrived at by different experimenters on the strength of materials. It scarcely ever occurs that a material is presented to us, either for experiment or for application to a practical use, in which the particles are free from great mutual strains. Processes have already been pointed out by which we may at pleasure produce certain peculiar strains of this kind. These or other processes producing somewhat similar strains are used in the manufacture of almost all materials. Thus, for instance, when malleable iron has received its final conformation by the process termed *cold swaging*, that is by hammering it till it is cold, the outer particles exist in a state of extreme compression, and the internal ones in a state of extreme tension. The same seems to be the case in cast iron when it is taken from the mould in which it has been cast. The outer portions have cooled first, and have therefore contracted while the inner ones still continued expanded by heat. The inner ones then contract as they subsequently cool, and thus they as it were pull the outer ones together. That is, in

the end, the outer ones are in a state of compression and the inner ones in the opposite condition.

The foregoing principles may serve to explain the true cause of an important fact obsered by Mr. Eaton Hodgkinson in his valuable researches in regard to the strength of cast iron (*Report of the British Association* for 1837, p. 362).* He found that, contrary to what had been previously supposed, a strain, however small in comparison to that which would occasion rupture, was sufficient to produce a set in the beams on which he experimented. Now this is just what should be expected in accordance with the principles which I have brought forward: for if, from some of the causes already pointed out, various parts of a beam previously to the application of an external force have been strained to the utmost, when, by the application of such force, however small, they are still farther displaced from their positions of relaxation, they must necessarily undergo a permanent alteration in their connexion with one another, an alteration permitted by the ductility of the material; or, in other words, the beam as a whole must take a set.

In accordance with this explanation of the fact observed by Mr. Hodgkinson, I do not think we are to conclude with him that "the maxim of loading bodies within the elastic limit has no foundation in nature." It appears to me that the defect of elasticity, which he has shewn to occur even with very slight strains, exists only when the strain is applied for the first time; or, in other words, that if a beam has already been acted on by a considerable strain, it may again be subjected to any smaller strain in the same direction without its taking a set. It will readily be seen, however, from Mr. Hodgkinson's experiments, that the term "elastic limit," as commonly employed, is entirely vague, and must tend to lead to erroneous results.

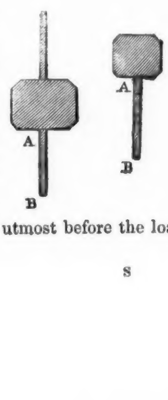
The considerations adduced seem to me to shew clearly that there really exist *two elastic limits* for any material, between which the displacements or deflexions, or what may in general be termed the changes of form, must be confined, if we wish to avoid giving the material a set; or, in the case of variable strains, if we wish to avoid giving it a continuous succession of sets which would gradually bring about its

* For farther information regarding Mr. Hodgkinson's views and experiments, see his communications in the *Transactions of the Sections of the British Association* for the years 1843 (p. 23) and 1844 (p. 25), and a work by him, entitled *Experimental Researches on the Strength and other Properties of Cast Iron.* 8vo. 1846.

destruction: that these two elastic limits are usually situated one on the one side, and the other on the opposite side of the position which the material assumes when subject to no external strain, though they may be both on the same side of this position of relaxation,* and that they may therefore with propriety be called the *superior* and the *inferior limit* of the change of form of the material for the particular arrangement which has been given to its particles; that these two limits are not *fixed* for any given material; but that if the change of form be continued beyond either limit, two new limits will, by means of an alteration in the arrangement of the particles of the material, be given to it in place of those which it previously possessed: and lastly, that the processes employed in the manufacture of materials are usually such as to place the two limits in close contiguity with one another, thus causing the material to take in the first instance a set from any strain however slight, while the interval which may afterwards exist between the two limits, and also as was before stated, the actual position assumed by each of them, is determined by the peculiar strains which are subsequently applied to the material.

The introduction of new, though necessary, elements into the consideration of the strength of materials may, on the one hand, seem annoying from rendering the investigations more complicated. On the other hand, their introduction will really have the effect of obviating difficulties, by removing erroneous modes of viewing the subject, and preventing contradictory or incongruous results from being obtained by theory and experiment. In all investigations, in fact, in which we desire to attain, or to approach nearly, to truth, we must take facts as they actually are, not as we might be tempted to wish them to be, for enabling us to dispense

* Thus, if the section of a beam be of some such form as that shewn in either of the accompanying figures, the one rib or the two ribs, as the case may be, being very weak in comparison to the thick part of the beam, it may readily occur that the two elastic limits of deflexion may be situated both on the same side of the position assumed by the beam when free from external force. For if the beam has been supported at its extremities and loaded at its middle till the rib AB has yielded by its ductility so as to make all its particles exert their utmost tension, and if the load be now gradually removed, the particles at B may come to be compressed to the utmost before the load has been entirely removed.



with examining processes which are somewhat concealed and intricate, but are not the less influential from their hidden character.

ON THE ELASTICITY AND STRENGTH OF SPIRAL SPRINGS, AND OF BARS SUBJECTED TO TORSION.

By James Thomson.

A Spiral Spring of the most usual kind consists of a long bar or wire, generally of a circular section, coiled up into the form of the thread of a screw.

For the purpose of attaining precision in speaking of such springs, the following definitions and preliminary explanations will be useful. The curve in which the centres of all the sections of the bar are situated may be called the *spiral axis* of the spring. This lies in the surface of a cylinder, the axis of which may be called the *longitudinal axis* of the spring. The angle which the spiral axis makes with a plane perpendicular to the longitudinal axis may be called the *inclination of the coil* or *of the spiral.* Each end of the bar is bent in such a way that the force applied to elongate or compress the spring may act in the longitudinal axis.* In what follows, unless the contrary be specified, the spring may be supposed to be suspended by one end, the force applied being one of tension produced by a weight hung at the lower end.

The Elasticity and Strength of Spiral Springs have not, so far as I am aware, been hitherto subjected to scientific investigation; and erroneous ideas are very prevalent on the subject, which are not unfrequently manifested in practice by the adoption of forms very different from those which would afford the greatest advantages. Having had occasion to construct some spiral springs which, in their elasticity, strength, and dimensions, should fulfil certain definite conditions, I was led to seek for principles to guide me in deter-

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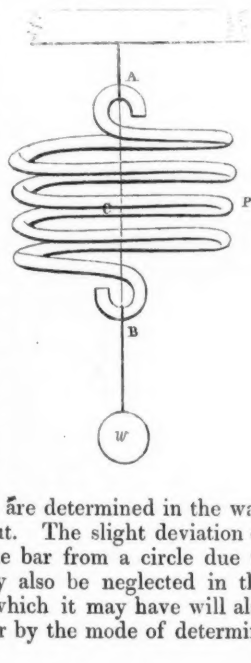
* In fact, even if, in the construction of the spring, this matter has not been attended to, and the ends of the bar forming the points of application of the opposite forces have not been placed in the longitudinal axis; immediately on a tensile force being applied, the spring will, if it be of considerable length, adjust itself so that the force will act in that axis, except in the neighbourhood of the two ends. At those parts the spring will be weaker than towards the middle; and, if the force be sufficient to induce a permanent alteration on its form, this change will commence by the ends assuming forms of greater resistance, the longitudinal axis approaching more nearly to the line of action of the forces.

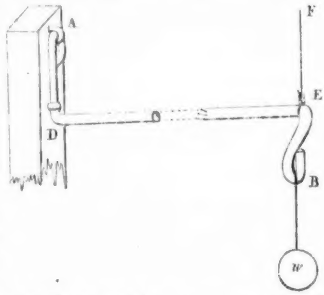
mining the forms and dimensions best adapted to accomplish the ends desired.

With this view, the first matter to be considered was the exact nature of the strains which act on the various parts of a spiral spring; and the whole subject became at once simple, as soon as I perceived that the only strain which produces a sensible effect is one of torsion, acting alike on every part of the coil. To render this clear, let us consider any section P of the bar, made by a plane passing through the longitudinal axis AB. Now, when a weight w is suspended at B, the forces transferred from one side to the other of this section consist of a couple whose force is w, and whose arm is the distance PC from the centre of the section at P to the line AB, together with a force w parallel to AB, tending to make the one side of the section slide upon the other in the direction of the longitudinal axis. The effect of this force must be extremely small, in fact evanescent, compared to that of the couple; and the force may therefore be neglected, especially when the coefficients for the elasticity and strength of the material are determined in the way which will be hereafter pointed out. The slight deviation of the above-mentioned section of the bar from a circle due to the inclination of the spiral, may also be neglected in the theory, as the minute influence which it may have will also be, in a great degree, corrected for by the mode of determining the coefficients.



Let now r be the radius of the bar; l the total length coiled up; a the radius of the coil, or the distance from the longitudinal to the spiral axis; w any weight which may be hung on the spring; e the elongation corresponding to that weight; and θ the angle through which a bar composed of the same substance as the spring, having its length and radius each unity, is twisted when subjected to a unit couple. This quantity θ may be called the coefficient of deflexion by torsion for the substance of which the bar is composed.

Let the spiral part of the bar be supposed to be straightened out, and let the bar be supposed to be placed with one end fixed at *AD*; and with a couple applied at the other extremity by means of a weight, equal to w, suspended at *B*, and a force in the line *EF* equal, parallel, and opposite to that of w; the arm of this couple being equal to a the radius of the coil.



Now it can be shewn that, in bars subjected to torsion, the angle of torsion is proportional to the length of the bar, to the couple applied, and inversely to the fourth power of the radius of the bar. Hence the angle of torsion in the present case, or the angle described by *EB*, in passing from its natural position to one of equilibrium with the couple wa is $= \theta \dfrac{lwa}{r^4}$.*

Hence, the space moved over by a point at a distance a from the centre is $= \theta \dfrac{lwa^2}{r^4}$. Now a little consideration will shew that this space must be equal to e, the elongation of the

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* As these principles regarding torsion have been laid down in various works on Mechanics and Engineering, I here take them for granted. The methods of deducing them have, however, been insufficient, as it has been, tacitly at least, assumed that all the elements in the section of a bar are free from strain when the bar as a whole is free from external strain; since it is assumed that when the bar is twisted to any extent less than that which would strain its circumference to the utmost, any equal elements of its section undergo strains proportional to the distances of the elements from the centre. It seldom occurs, however, that the real condition of a bar is in accordance with this assumption; for I have shewn in the preceding paper that the various particles of materials usually exist under great strains in opposite directions, which are in equilibrium with one another. The conclusions which had previously been derived are, however, in themselves correct; and in a note to the paper just referred to, I have supplied the step which was wanting in the proof by shewing that the angle of torsion of a bar, or in other words, its stiffness is not influenced by the presence or absence of internal opposing strains among its particles; although the case is very different with regard to the ultimate strength of the bar, which is materially altered by changes in those strains. From this it follows that the coefficient for the stiffness or deflexion of a bar composed of a given substance has but one value, while that for its ultimate strength may have various values. Of this more will be said in what follows.

spiral spring due to the application of the weight w. Hence we have

$$e = \theta \frac{lwa^2}{r^4} \dots\dots\dots\dots\dots (1).$$

This equation involves the conditions of the *elasticity*, or the *stiffness* of a spring as compared with its dimensions, and the substance of which it is composed. We will next proceed to those of its strength and its power; or, in other words, we will enter on considerations connected with the greatest weight which it can support, and the space through which it can be elongated without rupture or permanent alteration.

In addition to the notation given above:—Let W be the greatest weight which the spring, if its particles are all relaxed when it is not loaded, can bear without taking a set; E the greatest elongation, that namely which corresponds to W; μ the utmost couple producing torsion which can be resisted by a bar whose radius is unity, composed of the same substance as the spring, and having its particles at various distances from its centre free from mutual opposing strains when it, as a whole, is subject to no strain. In the preceding paper, "On the Strength of Materials," I shewed that the utmost couple which can be resisted by the bar will vary with the internal arrangement of the particles, its greatest value being $\frac{4}{3}$ and its least $\frac{2}{3}$ of its mean value, which, in the bar whose radius is unity, has just been denoted by μ. Any value of this couple, different from the mean one, adapted to a particular arrangement of the particles, may be denoted by μ', and the utmost weight and elongation in a spring having a similar arrangement may likewise be denoted by W' and E'. It will readily be seen that μ is to be regarded as the coefficient for any given substance, of the utmost strength of a cylindrical bar composed of it to resist torsion, when the particles of the bar have been so arranged that they may be all relaxed when the bar is free from external strain. It must, however, be remarked that μ would still represent the strength of the bar, even though there were *some* mutual opposing strains among the particles, provided that the particles at the circumference be relaxed when the bar is free from external strain; and that none of the internal particles exist under so great displacements from their positions of relaxation, as to occasion their being strained to the utmost sooner than the particles at the circumference, during the twisting of the bar.

If now, in the formula $L = \frac{1}{2}\pi\eta r^3$, for the strength of a bar when it is of its mean amount, which was proved in the paper

before referred to, we take $r = 1$, L will become what we have denoted by μ. Hence $\mu = \frac{1}{2}\pi\eta$, and the formula becomes L or the utmost couple $= \mu r^3$.

Again, since W is the utmost weight, and a the arm at which this acts, the utmost couple may also be expressed by Wa. Hence $Wa = \mu r^3$, and

$$W = \frac{\mu r^3}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The greatest elongation of which the spring can admit will be found by substituting in (1), W for w and E for e.

To justify us in making this substitution it must, however, be here remarked that, in ordinarily formed spiral springs, the elongations continue proportional to the weights added, even up to the very greatest that can be resisted. This fact I have myself observed by an experiment conducted with considerable care, and in which any deviation from the foregoing relation which may have existed was less than the inaccuracies of observation.*

Now, by making the substitution above indicated, of W for w, and E for e, in (1) we obtain

$$E = \theta \frac{l W a^2}{r^4} \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

which, by (2), may be put in the following form,

$$E = \theta\mu \frac{la}{r} \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

The equations (1) and (2) together with (3) or (4) involve the various circumstances connected with the elasticity and strength of ordinary spiral springs. For enabling us to determine, by means of these equations, the actual amounts of any of the variable quantities concerned, when a sufficient number of the variable quantities have been already fixed upon in accordance with the purposes to be effected; the constant coefficients θ and μ for the substance must be determined by experiment.

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* From this two distinct conclusions may be inferred: 1st. That the angle of torsion of a bar continues proportional to the applied couple as long as the arrangement of the particles remains unaltered; and, 2nd, That the alteration in the form of the spring by the increase of its angle of inclination and the consequent diminution of the radius of the coil does not produce a sensible effect. For, were there any deviation from the relation stated in the former proposition, this must in the experiment have been exactly counteracted by an effect of the change of form of the spring; which, it is clear, would have been a coincidence very unlikely to occur.

Without however knowing the actual amounts, we may, by interpreting the equations, arrive at many useful conclusions for the comparison of the properties of springs constructed of the same substance, but having various dimensions. From (1) we see that—

1st. If r the radius of the bar, and a that of the coil, be fixed; the elongation produced by any weight w will be proportional to l the length rolled up to form the coil.

2nd. If a bar or wire of a certain length and radius be given to form a spring; the elongation produced by a certain weight w will be proportional to the square of the radius which we may adopt for the coil.

3rd. If the radius of the bar be fixed, and the length of the spring when closed so that the coils may touch one another, or what is the same, the number of coils be also fixed; l must be proportional to a; and therefore the elongation due to a weight w will be proportional to the third power of the radius which we may adopt for the coil.

4th. If the length of the bar and the radius of the coil be fixed, the elongation due to a weight w, will be inversely proportional to the fourth power of the radius of the bar which we may adopt.

5th. With a given weight of metal and a given radius of the coil; the elongation, due to a weight w, will be proportional to l^3, or inversely to r^6, since l must be proportional to $\frac{1}{r^2}$.

From (4) we see that the ultimate elongation is,

1st. Proportional to the length of the bar, if the radius of the bar and that of the coil be fixed.

2nd. Proportional to the radius of the coil, if the length and the radius of the bar be fixed.

3rd. Inversely proportional to the radius of the bar, if the length of the bar and the radius of the coil be fixed.

From (2) we perceive that the absolute strength of the spring, being independent of the length, is proportional to the third power of the radius of the bar, if the radius of the coil be fixed; and that it is inversely proportional to the radius adopted for the coil, if the radius of the bar be fixed.

By combining (2) and (4) we arrive at the interesting conclusion that the "*resilience*" of a spiral spring, that is, the total quantity of work which can be stored up in it, is independent of the form or proportions of the spring, and is

simply proportional to the quantity of metal contained in the coil. For, since the weights producing any elongations are proportional to those elongations, it follows that the resilience is $= \frac{1}{2} W.E.$

Hence, by (2) and (4), we find that the resilience is $= \frac{1}{2}\theta\mu^2 lr^2$, which, since θ and μ are constant, is proportional to the volume of the coil or to the weight of metal composing it.

Many other relations might be deduced in similar ways, but those already pointed out will suffice, as others will be readily perceived when they may be wanted, by properly interrogating the formulas.

For determining the values of μ and θ for iron wire such as is commonly used for making spiral springs (called Charcoal Spring Wire); a spring constructed of this material was subjected to careful measurement and experiment, and the following data were obtained:

$$\text{Dimensions of the spring} \begin{cases} r = \cdot 0923 \text{ inches,} \\ l = 215\cdot 6 \text{ inches,} \\ a = 1\cdot 315 \text{ inches.} \end{cases}$$

When the spring was successively loaded with weights, each four pounds heavier than the one before; it was found that 56 pounds was the weight which just commenced to produce a permanent elongation, and the elongation corresponding to this weight was observed to be 16·9 inches. By the method which had been employed for bending the wire into the spiral form, and for separating the coils so that they might not press on one another when the spring was unloaded, the wire had been put into the condition which it would have received by having been twisted beyond the original elastic limit, that condition, namely, in which nearly all the particles in its section would come to be strained to the utmost at the same time. Hence, according to the notation which has been adopted, this ultimate weight and elongation must be denoted by W' and E'; which, by the principles given in a former paper, already referred to, will be such that

W' shall be equal to, or rather less than, $\frac{4}{3} W$,

and E' equal to, or rather less than, $\frac{4}{3} E$.*

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* If we could be sure that the twisting of the wire, before the experiment, had been sufficiently great, it might be stated with certainty that $W' = \frac{4}{3} W$, and $E' = \frac{4}{3} E$. The uncertainty in this respect, though it cannot much affect the resulting value of μ, is the greatest to which the present determination of the coefficient μ is subject.

By taking W', μ', and E', for W, μ, and E in equations (2) and (3), we get

$$W' = \frac{\mu' r^3}{a}, \quad \text{and } E' = \theta \frac{lW'a^2}{r^4};$$

From which, by the foregoing experimental data, we obtain

$$\theta = 0{\cdot}000{,}000{,}059^* \quad \text{and } \mu' = 94{,}000.$$

But μ' is equal to, or rather less than, $\frac{4}{3}\mu$. Hence

$$\mu = 70{,}000, \text{ or rather more.}$$

The values here assigned to θ and μ will, I think, be found quite sufficiently accurate for all practical purposes; especially as the metal, in other cases, cannot be assumed to be of exactly the same quality as that used in the present one. The experiment from which these values have been deduced is the only one I have as yet been able to make; I hope, however, as soon as circumstances may permit, to carry out a consecutive series of comparative experiments with springs of various dimensions and forms and of several different kinds of metal, and also to make corresponding experiments by subjecting to direct torsion, bars similar to those forming the springs. In this way would be found the amount by which the coefficients may vary in accordance with different circumstances; and, in applying the formulas afterwards to any particular case, we should be able to choose such values of the coefficients as might appear most suitable; or at least we should know what degree of dependence ought to be placed in results obtained by applying the formulas to such cases. If a spring having the radius of the bar very small compared to that of the coil, and having the angle of its spiral as small as possible, were used, there can be no doubt but that the coefficients so obtained would agree very closely with those deduced from experiment by direct torsion.

Before concluding, I shall now merely remark that, according to the principles already given, spiral springs should always be made of bars of circular section, since such bars

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* It may here be observed that for the determination of θ the extreme weight and elongation W' and E' should not have been used, unless it had been found that these would lead to the same results as any smaller weight and elongation w and e. It was however found, as has been already stated, that the elongations were throughout proportional to the weights applied; and therefore it is a matter of indifference what weight and corresponding elongation we adopt for the determination of θ.

have a much greater resilience* when subjected to torsion than others whose section is different. Thus we see that the rectangular section which has been frequently adopted in large spiral springs for railway carriage buffers is very disadvantageous. It has probably been derived from the idea that the bar is subjected to a transverse strain; an idea which, however erroneous, is the one that usually presents itself to persons considering in a cursory manner the action of spiral springs.

ON THE MATHEMATICAL THEORY OF ELECTRICITY IN EQUILIBRIUM.

IV.—GEOMETRICAL INVESTIGATIONS REGARDING SPHERICAL CONDUCTORS.

By William Thomson.

[*Continued from* page 148.]

Insulated Sphere subjected to the influence of an Electrical Point.

1. A conducting sphere placed in the neighbourhood of an electrified body must necessarily become itself electric, even if it were previously uncharged; since (Law III.) the entire resultant force at any point within it must vanish, and consequently there must be a distribution of electricity on its surface which will for internal points balance the force resulting from the external electrified body. If the sphere, being insulated, be previously charged with a given quantity of electricity, the whole amount will (I. § 11) remain unaltered by the electrical influence, but its distribution cannot be uniform, since in that case it would exert no force on an internal point, and there would remain the unbalanced resultant due to the external body. In what follows, it will be proved that the conditions are satisfied by a certain assumed distribution of electricity in each instance; but the proposition that no other distribution can satisfy the conditions, which is merely a case of a general theorem referred to above (II. § 13) will not be specially demonstrated with reference to the particular problems; although we shall have to assume its truth

* The meaning of the term "*resilience*" was before stated to be the quantity of work which can be stored up in the material, by giving to it its utmost change of form.

when a certain distribution which is proved synthetically to satisfy the conditions is asserted to be the unique solution of the problem.

Attraction of a Spherical Surface of which the density varies inversely as the cube of the distance from a given point.

2. Let us first consider the case in which the given point S and the attracted point P are separated by the spherical surface. The two figures represent the varieties of this case in which, the point S being without the sphere, P is within; and, S being within, the attracted point is external. The same demonstration is applicable literally with reference to the two figures; but, for avoiding the consideration of negative quantities, some of the expressions may be conveniently modified to suit the second figure. In such instances the two expressions are given in a double line, the upper being that which is most convenient for the first figure, and the lower for the second.

Let the radius of the sphere be denoted by a, and let f be the distance of S from C, the centre of the sphere (not represented in the figures).

Join SP and take T in this line (or its continuation) so that

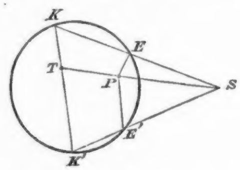
$$\left.\begin{array}{ll}\text{(fig. 1)} & SP.ST = f^2 - a^2 \\ \text{(fig. 2)} & SP.TS = a^2 - f^2\end{array}\right\} \quad \ldots\ldots\ldots\ldots \text{(1)*}.$$

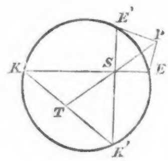
Through T draw any line cutting the spherical surface at K, K'. Join SK, SK', and let the lines so drawn cut the spherical surface again in E, E'.

Let the whole spherical surface be divided into pairs of opposite elements with reference to the point T. Let K

* If, in geometrical investigations in which diagrams are referred to, the distinction of *positive* and *negative* quantities be observed, the order of the letters expressing a straight line will determine the algebraic sign of the *quantity* denoted: thus we should have, universally, if A, B be the extremities of a straight line; $AB = -BA$, each member of this equation being positive or negative according to the conventional direction in which positive quantities are estimated. In the present instance, lengths measured along the line SP in the direction from S towards P, or in corresponding directions in the continuation of this line on either side, are, in both figures, considered as positive. Hence, in the first figure ST will be positive; but when f is less than a, ST must be negative on account of the equation $SP.ST = f^2 - a^2$. Hence the second figure represents this case; and, if we wish to express the circumstances without the use of negative quantities, we must change the signs of both members of the equation, and substitute for the positive quantity $-ST$ its equivalent TS, so that we have $SP.TS = a^2 - f^2$, as the most convenient form of the expression, when reference is made to the second figure. See above (Symbolical Geometry, § 4, in Vol. I. of this *Journal*), where the principles of interpretation of the sign $-$ in geometry are laid down by Sir William Hamilton.

and K' be a pair of such elements situated at the extremities of the chord KK', and subtending the solid angle ω at the point T; and let elements E and E' be taken subtending at S the same solid angles respectively as the elements K and K'. By this means we may divide the whole spherical surface into pairs of conjugate elements, E, E', since it is easily seen that when we have taken every pair of elements, K, K', the whole surface will have been exhausted, without repetition, by the deduced elements, E, E' Hence the





attraction on P will be the final resultant of the attractions of all the pairs of elements, E, E'.

Now if ρ be the electrical density at E, and if F denote the attraction of the element E on P, we have

$$F = \frac{\rho . E}{EP^2}.$$

According to the given law of density we shall have

$$\rho = \frac{\lambda}{SE^3},$$

where λ is a constant. Again, since SEK is equally inclined to the spherical surface at the two points of intersection, we have (III. §§ 10, 11)

$$E = \frac{SE^2}{SK^2} . K = \frac{SE^2}{SK^2} . \frac{2a\omega . TK^2}{KK'};$$

and hence

$$F = \frac{\frac{\lambda}{SE^3} . \frac{SE^2}{SK^2} . \frac{2a\omega . TK^2}{KK'}}{EP^2} = \lambda . \frac{2a}{KK'} . \frac{TK^2}{SE . SK^2 . EP^2} . \omega .$$

Now, by considering the great circle in which the sphere is cut by a plane through the line SK, we find that

$$\left.\begin{array}{ll}\text{(fig. 1)} & SK . SE = f^2 - a^2 \\ \text{(fig. 2)} & KS . SE = a^2 - f^2\end{array}\right\} \ldots\ldots\ldots\ldots (2),$$

and hence $SK . SE = SP . ST$, from which we infer that the

triangles KST, PSE are similar; so that $TK:SK::PE:SP$. Hence

$$\frac{TK^2}{SK^2.PE^2} = \frac{1}{SP^2},$$

and the expression for F becomes

$$F = \lambda \cdot \frac{2a}{KK'} \cdot \frac{1}{SE.SP^2} \cdot \omega \dots\dots\dots (3).$$

Modifying this by (2) we have

$$\left.\begin{aligned} &(\text{fig. } 1) \quad F = \lambda. \frac{2a}{KK'} \cdot \frac{\omega}{(f^2 - a^2)\, SP^2} \cdot SK \\ &(\text{fig. } 2) \quad F = \lambda \frac{2a}{KK'} \cdot \frac{\omega}{(a^2 - f^2)\, SP^2} \cdot KS \end{aligned}\right\} \dots (4).$$

Similarly, if F' denote the attraction of E' on P, we have

$$(\text{fig. } 1) \quad F' = \lambda \frac{2a}{KK'} \cdot \frac{\omega}{(f^2 - a^2)\, SP^2} \cdot SK',$$

$$(\text{fig. } 2) \quad F' = \lambda \frac{2a}{KK'} \cdot \frac{\omega}{(a^2 - f^2)\, SP^2} \cdot K'S.$$

Now in the triangles which have been shewn to be similar, the angles TKS, EPS are equal; and the same may be proved of the angles $K'ST$, PSE'. Hence the two sides SK, SK' of the triangle KSK' are inclined to the third at the same angles as those between the line PS and directions PE, PE' of the two forces on the point P; and the sides SK, SK' are to one another as the forces, F, F', in the directions PE, PE'. It follows, by "the triangle of forces" that the resultant of F and F' is along PS, and that it bears to the component forces the same ratios as the side KK' of the triangle bears to the other two sides. Hence the resultant force due to the two elements E and E', on the point P, is towards S, and is equal to

$$\lambda \cdot \frac{2a}{KK'} \cdot \frac{\omega}{(f^2 \sim a^2) \cdot SP^2} \cdot KK', \quad \text{or} \quad \frac{\lambda.2a.\omega}{(f^2 \sim a^2)\, SP^2}.$$

The total resultant force will consequently be towards S; and we find, by summation (III. § 8) for its magnitude,

$$\frac{\lambda.4\pi a}{(f^2 \sim a^2)\, SP^2}.$$

Hence we infer that the resultant force at any point P, separated from S by the spherical surface, is the same as if

a quantity of matter equal to $\dfrac{\lambda.4\pi a}{f^2 \sim a^2}$ were concentrated at the point S.

3. To find the attraction when S and P are either both without or both within the spherical surface.

Take in CS (fig. 3), or in CS produced through S (fig. 4), a point S_1, such that

$$CS.CS_1 = a^2.$$

Then, by a well known geometrical theorem (See III., Note on § 12) if E be any point on the spherical surface, we have

$$\frac{SE}{S_1E} = \frac{f}{a}.$$

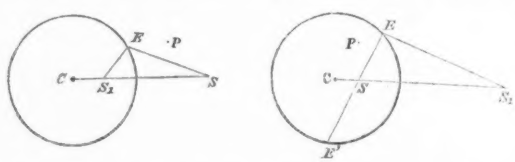
Hence we have $$\frac{\lambda}{SE^3} = \frac{\lambda a^3}{f^3 . SE_1^3},$$

Hence, ρ being the electrical density at E, we have

$$\rho = \frac{\frac{\lambda a^3}{f^3}}{S_1E^3} = \frac{\lambda_1}{S_1E^3},$$

if $$\lambda_1 = \frac{\lambda a^3}{f^3}.$$

Hence, by the investigation in the preceding paragraph, the



attraction on P is towards S_1, and is the same as if a quantity of matter equal to $\dfrac{\lambda_1 . 4\pi a}{f_1^2 \sim a^2}$ were concentrated at that point; f_1 being taken to denote CS_1. If for f_1 and λ_1 we substitute their values, $\dfrac{a^2}{f}$ and $\dfrac{\lambda a^3}{f^3}$, we have the modified expression

$$\frac{\lambda \frac{a}{f} . 4\pi a}{a^2 \sim f^2}$$

for the quantity of matter which we must conceive to be collected at S_1.

4. PROP. If a spherical surface be electrified in such a way that the electrical density varies inversely as the cube of the distance from an internal point S (fig. 4), or from the corresponding external point S_1, it will attract any external point, as if its whole mass were concentrated at S, and any internal point as if a quantity of matter greater than the whole mass in the ratio of a to f were concentrated at S_1.

Let the density at E be denoted, as before, by $\dfrac{\lambda}{SE^3}$. Then, if we consider two opposite elements at E and E' which subtend a solid angle ω at the point S, the areas of these elements being (III. § 11) $\dfrac{\omega.2a.SE^2}{EE'}$ and $\dfrac{\omega.2a.SE'^2}{EE'}$, the quantity of electricity which they possess will be

$$\frac{\lambda.2a.\omega}{EE'}\left(\frac{1}{SE}+\frac{1}{SE'}\right) \text{ or } \frac{\lambda.2a.\omega}{SE.SE'}.$$

Now $SE\,SE'$ is constant (Euc. III. 35) and its value is a^2-f^2. Hence, by summation, we find for the total quantity of electricity on the spherical surface

$$\frac{\lambda.4\pi a}{a^2-f^2}.$$

Hence, if this be denoted by m, the expressions in the preceding paragraphs, for the quantities of electricity which we must suppose to be concentrated at the point S or S_1, according as P is without or within the spherical surface, become respectively

$$m, \text{ and } \frac{a}{f}m. \qquad \text{Q. E. D.}$$

Application of the preceding Theorems to the Problem of Electrical Influence.

5. PROB. To find the electrical density at any point of an insulated conducting sphere (radius a) charged with a quantity Q (either positive, or negative, or zero) of electricity, and placed with its centre at a given distance f from an electrical point M possessing m units of electricity.

If the expression for the electrical density at any point E of the surface be

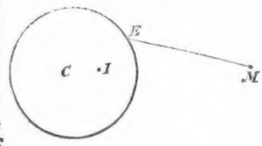
$$\rho=\frac{\lambda}{ME^3}+k \ldots\ldots\ldots\ldots\ldots (a),$$

λ and k being constants; the force exerted by the electrified surface on any internal point will be the same as if the

constant distribution k, which (III. § 3) exerts no force on an internal point, were removed; and therefore (IV. § 2) will be the same as if a quantity of matter equal to $\frac{\lambda.4\pi a}{f^2 - a^2}$ were collected at the point M. Hence, if the condition

$$\frac{\lambda.4\pi a}{f^2 - a^2} = -m \ldots\ldots\ldots\ldots\ldots (b)$$

be satisfied, the total attraction on an internal point, due to the electrified surface and to the influencing point, will vanish. Hence this distribution satisfies the condition of equilibrium (II. § 12); and to complete the solution of the proposed problem it only remains to determine the quantity k, so that the total quantity of electricity on the surface may have the given value Q. Now (§ 4) the total mass of the distribution, depending on the term $\frac{\lambda}{ME^3}$ in the expression for the density, since M is an *external* point, is equal to



$$\frac{a}{f} \cdot \frac{\lambda.4\pi a}{f^2 - a^2}.$$

Hence, adding $4\pi a^2 k$, the quantity depending on the constant term k, we obtain the entire quantity, which must be equal to Q; and we therefore have the equation

$$\frac{a}{f} \cdot \frac{\lambda.4\pi a}{f^2 - a^2} + 4\pi a^2 k = Q \ldots\ldots\ldots\ldots (c).$$

From equations (b) and (c) we deduce

$$\lambda = -\frac{(f^2 - a^2)\,m}{4\pi a} \quad \text{and} \quad k = \frac{Q + \frac{a}{f}\,m}{4\pi a^2}.$$

Hence, by substituting in (a), we have

$$\rho = -\frac{(f^2 - a^2)\,m}{4\pi a} \cdot \frac{1}{ME^3} + \frac{Q + \frac{a}{f}\,m}{4\pi a^2} \ldots . (A),$$

as the expression of the required distribution of electricity. This agrees with the result obtained by Poisson, by means of an investigation in which the analysis known as that of "Laplace's coefficients," is employed.

6. To find the attraction exerted by the electrified conductor on any external point.

We may consider separately the distributions corresponding to the constant and the variable term in the expression for the electrical density at any point of the surface. The attraction of the first of these on an external point is (III. § 12) the same as if its whole mass were collected at the centre of the sphere: the attraction of the second on an external point is (IV. § 4) the same as if its whole mass were collected at an interior point I, taken in MC so that $MI.MC = a^2$. Hence, according to the investigation in the preceding paragraph, we infer that the conductor attracts any external point with the same force as would be produced by quantities $Q + \frac{a}{f}m$, and $-\frac{a}{f}m$ of electricity, concentrated at the points C and I respectively.

Cor. The resultant force at an external point infinitely near the surface is in the direction of the normal, and is equal to $4\pi\rho$, if ρ be the electrical density of the surface, in the neighbourhood.

7. To find the mutual attraction or repulsion between the influencing point, M, and the conducting sphere.

According to what precedes, the required attraction or repulsion will be the entire force exerted upon m units of electricity at the point M, by $Q + \frac{a}{f}m$ at C and $-\frac{a}{f}m$ at a point I, taken in CM, at a distance $\frac{a^2}{f}$ from C. Hence, if the required *attraction* be denoted by F (a quantity which will be negative if the actual force be of *repulsion*), we have

$$F = -\frac{\left(Q + \frac{a}{f}m\right)m}{f^2} + \frac{\frac{a}{f}m.m}{\left(f - \frac{a^2}{f}\right)^2} \ldots\ldots\ldots (B),$$

$$= \frac{a\left\{f^4 - (f^2 - a^2)^2\right\}m^2 - f(f^2 - a^2)^2 Qm}{f^3(f^2 - a^2)^2},$$

or

$$F = \frac{a(2f^2a^2 - a^4)m^2 - f(f^2 - a^2)^2 Qm}{f^3(f^2 - a^2)^2} \ldots (C).$$

Cor. 1. If Q be zero or negative, the value of F is necessarily positive, since f must be greater than a; and therefore

there is a force of attraction between the influencing point and the conducting sphere, whatever be the distance between them.

COR. 2. If Q be positive, then for sufficiently large values of f, F is negative, while for values nearly equal to a, F is positive. Hence if an electrical point be brought into the neighbourhood of a similarly charged insulated sphere, and if it be held at a great distance, the mutual action will be repulsive; if it then be gradually moved towards the sphere, the repulsion, which will at first increase, will, after attaining a maximum value, begin to diminish till the electrical point is moved up to a certain distance where there will be no force either of attraction or repulsion; if it be brought still nearer to the conductor, the action will become attractive and will continually augment as the distance is diminished.

If the value of Q be positive, and sufficiently great, a spark will be produced between the nearest part of the conductor and the influencing point, before the force becomes changed from repulsion to attraction.

St. Peter's College, July 7, 1848.

ON DIFFERENTIATION WITH FRACTIONAL INDICES, AND ON GENERAL DIFFERENTIATION.

By the Rev. WILLIAM CENTER.

THE definite integral $\int_0^\infty \epsilon^{-a} a^{m-1}\, da = \Gamma m$, is due to Euler, who, seeing its importance, devoted several memoirs to its investigation. The symbol Γ is due to Legendre, as also the functional equation $\Gamma(1+m) = m\,\Gamma m$, with the limit $m > 0$, the same as in the Eulerian integral. From this integral proceeds the well-known form

$$\frac{1}{x^m} = \frac{1}{\Gamma m} \int_0^\infty \epsilon^{-ax} a^{m-1}\, da,$$

having the limits $m > 0$ and $x > 0$. By differentiating θ times relatively to x, we have

$$\left(\frac{d}{dx}\right)^\theta \frac{1}{x^m} = \frac{(-1)^\theta}{\Gamma m} \int_0^\infty \epsilon^{-ax} a^{m+\theta-1}\, da$$

limits $m > 0$ and $(m+\theta) > 0$, $\quad = (-1)^\theta \dfrac{\Gamma(m+\theta)}{\Gamma m} \cdot \dfrac{1}{x^{m+\theta}}$.

On examining this expression, Liouville was the first to observe that it must necessarily extend to fractional values of θ the index of differentiation, in order that the equation might embrace all the values of m and θ within the amplitude of the function $\Gamma(m+\theta)$. Hence he proposed it as a formula of differentiation most likely general, and verified it very extensively. Thus one case of general differentiation, limited only by the necessary condition $m > 0$ and $(m + \theta) > 0$, was fairly deduced in continuity with the common Calculus. But the immediate generalization of this differential expression into $\left(\frac{d}{dx}\right)^\theta x^m = (-1)^\theta \frac{\Gamma(-m+\theta)}{\Gamma(1-m)} x^{m-\theta}$, included the passage *per saltum* over the proper limits $m > 0$ and $(m + \theta) > 0$ of the definite integral from which the primary derivation is made; and apparently involved such an inconsistency in logical deduction, as induced Dr. Peacock, in his well known *Report on Analysis*, to propose a distinct system, and to give the formula

$$\left(\frac{d}{dx}\right)^\theta x^m = \frac{\Gamma(1+m)}{\Gamma(1+m-\theta)} x^{m-\theta}.$$

The irreconcilable character of these two proposed formula disposed Professor De Morgan (*Diff. Cal.*, p. 599) to admit them both as particular cases of a yet more general system, but to exclude either from any pretension to generality.

The subject has been lately resumed by Professor Kelland (*Transactions of the Royal Society of Edinburgh*, 1847), who has pushed his investigations in the subject of general differentiation to a very great extent, on the assumption of the generality of Liouville's formula. The subject, therefore, demands a rigorous investigation.

§ 1. The problem, which the present state of the subject requires to be solved, seems to be fully embraced in the following proposition :—

"Granting the differential operation

$$\left(\frac{d}{dx}\right)^\theta \frac{1}{x^m} = \frac{(-1)^\theta}{\Gamma(m)} \int_0^\infty \epsilon^{-ax} a^{m+\theta-1} da = (-1)^\theta \frac{\Gamma(m+\theta)}{\Gamma(m)} \cdot \frac{1}{x^{m+\theta}} \ldots . (\mathrm{A}),$$

within the limits $\qquad m > 0, (m + \theta) > 0$;

by a continuous deductive process, that shall always lie within the proper limit $C > 0$ of Legendre's functional equation $\Gamma(1 + C) = C\Gamma(C)$, . . ($L$) to assign the differential coefficient

of $\left(\frac{d}{dx}\right)^{\mu} x^m$ for all values of m, positive or negative, whole or fractional; while μ, the index of operation, is a proper fraction, either positive or negative."

The method of the separation of the symbols of operation and quantity affords the means of giving a complete solution to this problem. Of this method we require only at present the *laws of indices.*

Case 1. When m is any positive number equal to $(n+p)$, n being the integer and p the proper fractional part of m.

Since n is a whole nnmber, the operation indicated by $\left(\frac{d}{dx}\right)^{-n}$ is one purely of common sucessive integration, hence

$$\left(\frac{d}{dx}\right)^{-n} x^p = \frac{x^{n+p}}{(p+1)(p+2)\ldots(p+n)} = \frac{\Gamma(1+p)}{\Gamma(1+p+n)}\, x^{n+p}$$

(expressing the *factorial* by Legendre's function); therefore

$$\left(\frac{d}{dx}\right)^{-n} x^p = \frac{\Gamma(1+p)}{\Gamma(1+m)}\, x^m \ldots\ldots\ldots\ldots (1),$$

hence $$x^m = \frac{\Gamma(1+m)}{\Gamma(1+p)} \left(\frac{d}{dx}\right)^{-n} x^p.$$

Operate on both sides with $\left(\frac{d}{dx}\right)$, and we have

$$\begin{aligned}\left(\frac{d}{dx}\right) x^m &= \frac{\Gamma(1+m)}{\Gamma(1+p)} \left(\frac{d}{dx}\right)^{-n+1} x^p \\ &= \frac{\Gamma(1+m)}{\Gamma(1+p)} \left(\frac{d}{dx}\right)^{-n} \left\{\left(\frac{d}{dx}\right) x^p\right\} \\ &= \frac{\Gamma(1+m)}{\Gamma(1+p)} \left(\frac{d}{dx}\right)^{-n} \{p x^{p-1}\} \\ &= p\, \frac{\Gamma(1+m)}{\Gamma(1+p)} \left(\frac{d}{dx}\right)^{-n} x^{p-1} \ldots\ldots\ldots\ldots (2)\end{aligned}$$

Since p is a proper fraction, put $q = 1 - p$; hence $p - 1 = -p$; substitute in (2), therefore

$$\left(\frac{d}{dx}\right) x^m = p\, \frac{\Gamma(1+m)}{\Gamma(1+p)} \left(\frac{d}{dx}\right)^{-n} x^{-q} \ldots\ldots\ldots (3).$$

(1) Let μ be positive, and operate on (3) with $\left(\frac{d}{dx}\right)^{\mu}$,

$$\therefore \left(\frac{d}{dx}\right)^{\mu+1} x^m = \frac{p\,\Gamma(1+m)}{\Gamma(1+p)} \left(\frac{d}{dx}\right)^{-n+\mu} x^{-q}$$

$$= \frac{p\,\Gamma(1+m)}{\Gamma(1+p)} \left(\frac{d}{dx}\right)^{-n} \left\{\left(\frac{d}{dx}\right)^{\mu} x^{-q}\right\}$$

by (A),
$$= \frac{p\,\Gamma(1+m)}{\Gamma(1+p)} \left(\frac{d}{dx}\right)^{-n} \left\{(-1)^{\mu} \frac{\Gamma(q+\mu)}{\Gamma(q)} x^{-q-\mu}\right\}$$

$$\therefore \left(\frac{d}{dx}\right)^{\mu+1} x^m = (-1)^{\mu} \frac{p\,\Gamma(q+\mu)\,\Gamma(1+m)}{\Gamma(q)\,\Gamma(1+p)} \left(\frac{d}{dx}\right)^{-n} x^{-q-\mu} \,..\, (4).$$

Operate on (4) with $\left(\frac{d}{dx}\right)^{-1}$, and we have

$$\left(\frac{d}{dx}\right)^{\mu} x^m = (-1)^{\mu} \frac{p\,\Gamma(q+\mu)\,\Gamma(1+m)}{\Gamma(q)\,\Gamma(1+p)} \left(\frac{d}{dx}\right)^{-n-1} x^{-q-\mu}$$

$$= (-1)^{\mu} \frac{p\,\Gamma(q+\mu)\,\Gamma(1+m)}{\Gamma(q)\,\Gamma(1+p)} \left(\frac{d}{dx}\right)^{-n} \left\{\left(\frac{d}{dx}\right)^{-1} x^{-q-\mu}\right\}$$

$$= (-1)^{\mu} \frac{p\,\Gamma(q+\mu)\,\Gamma(1+m)}{\Gamma(q)\,\Gamma(1+p)} \left(\frac{d}{dx}\right)^{-n} \left\{\frac{x^{1-q-\mu}}{1-q-\mu}\right\}$$

$$= (-1)^{\mu} \frac{p\,\Gamma(q+\mu)\,\Gamma(1+m)}{(1-q-\mu)\,\Gamma(q)\,\Gamma(1+p)} \left(\frac{d}{dx}\right)^{-n} x^{1-q-\mu}$$

replace p, ..
$$= (-1)^{\mu} \frac{p\,\Gamma(1-p+\mu)\,\Gamma(1+m)}{(p-\mu)\,\Gamma(1-p)\,\Gamma(1+p)} \left(\frac{d}{dx}\right)^{-n} x^{p-\mu}$$

by (L), ...
$$= (-1)^{\mu} \frac{p\,\Gamma(2-p+\mu)\,\Gamma(1+m)}{(p-\mu)(1-p+\mu)\,\Gamma(1-p)\,\Gamma(1+p)} \left(\frac{d}{dx}\right)^{-n} x^{p-\mu} \quad ......(5).$$

But since n is a whole number, we have, as in (1),

$$\left(\frac{d}{dx}\right)^{-n} x^{p-\mu} = \frac{\Gamma(1+p-\mu)}{\Gamma(1+p+n-\mu)} x^{n+p-\mu}$$

$$= \frac{\Gamma(1+p-\mu)}{\Gamma(1+m-\mu)} x^{m-\mu} \,..........\, (6).$$

By substituting (6) in (5) we have finally,

$$\left(\frac{d}{dx}\right)^{\mu} x^m = (-1)^{\mu} \frac{p\,\Gamma(2-p+\mu)\,\Gamma(1+p-\mu)\,\Gamma(1+m)}{(p-\mu)(1-p+\mu)\,\Gamma(1-p)\,\Gamma(1+p)\,\Gamma(1+m-\mu)}\, x^{m-\mu} \quad \ldots\ldots(\mathrm{B}).$$

(2) By changing the sign of μ the index of operation in (B), we immediately perceive that (B) will still lie within the limits of Legendre's function; so that (B) is the differential coefficient $\left(\frac{d}{dx}\right)^{\mu} x^m$, whether μ be positive or negative, provided it be fractional.

By putting B_μ for the coefficient of $x^{m-\mu}$ in (B), we may express (*B*) in the following forms according as μ is positive or negative:

$$\left(\frac{d}{dx}\right)^{\mu} x^m = B_\mu x^{m-\mu} \quad \ldots\ldots\ldots\ldots (7),$$

$$\left(\frac{d}{dx}\right)^{-\mu} x^m = B_{-\mu} x^{m+\mu} \ldots\ldots\ldots\ldots (8).$$

The abridged forms will be very convenient for future reference.

Cor. 1. When m is a whole number, put $p = 0$ in (B); hence $m = n$, therefore

$$\left(\frac{d}{dx}\right)^{\mu} x^n = (-1)^{\mu}. \frac{0.\Gamma(2+\mu)\,\Gamma(1-\mu)\,\Gamma(1+n)}{-\mu(1+\mu)\,\Gamma(1+n-\mu)}\, x^{n-\mu},$$

by (L),

$$= (-1)^{\mu}. \frac{0.\Gamma(1+\mu)\,\Gamma(1-\mu)\,\Gamma(1+n)}{-\mu.\Gamma(1+n-\mu)}\, x^{n-\mu} \quad \ldots. (9).$$

Cor. 2. When m is a proper fraction, put $n = 0$ in (B); hence $m = p$, therefore

$$\left(\frac{d}{dx}\right)^{\mu} x^p = (-1)^{\mu}. \frac{p\,\Gamma(2-p+\mu)\,\Gamma(1+p-\mu)\,\Gamma(1+p)}{(p-\mu)(1-p+\mu)\,\Gamma(1-p)\,\Gamma(1+p)\,\Gamma(1+p-\mu)}\, x^{p-\mu}$$

$$= (-1)^{\mu}. \frac{p.\Gamma(2-p+\mu)}{(p-\mu)(1-p+\mu)\,\Gamma(1-p)}\, x^{p-\mu} \ldots\ldots(10).$$

Case 2. When m is negative, we have at once, by (A),

$$\left(\frac{d}{dx}\right)^{\mu} x^{-m} = (-1)^{\mu} \frac{\Gamma(m+\mu)}{\Gamma(m)}\, x^{-m-\mu},$$

by (L),

$$= (-1)^{\mu} \frac{\Gamma(1+m+\mu)}{(\mu+m)\,\Gamma(m)}\, x^{-m-\mu} \ldots\ldots(\mathrm{C}).$$

Since μ is a proper fraction, we instantly perceive that (C) is within the limits of Legendre's function, whether μ be positive or negative.

By putting C_μ for the coefficient of $x^{-m-\mu}$ in (C), we have the following forms according as μ, the index of operation, is positive or negative. Thus

$$\left(\frac{d}{dx}\right)^{\mu} x^{-m} = C_\mu . x^{-m-\mu} \;\ldots\ldots\ldots\ldots (11),$$

and
$$\left(\frac{d}{dx}\right)^{-\mu} x^{-m} = C_{-\mu} . x^{-m+\mu} \ldots\ldots\ldots\ldots (12).$$

It thus appears that the formulæ (B) and (C) embrace the complete solution of the problem of fractional differentiation for all values of μ, the index of operation, from $\mu = 1$ to $\mu = -1$, having been deduced by a continuous process, which lies always within the proper limits of Legendre's function.

§ 2. *On General Differentiation.*

(1) If β be any positive whole number, while μ remains a positive proper fraction, as before, put $\lambda = \beta + \mu$; then λ will express *any* positive number, and, by the *law of indices*,

$$\left(\frac{d}{dx}\right)^{\lambda} = \left(\frac{d}{dx}\right)^{\beta+\mu}.$$

Hence
$$\left(\frac{d}{dx}\right)^{\lambda} x^m = \left(\frac{d}{dx}\right)^{\beta+\mu} x^m$$

$$= \left(\frac{d}{dx}\right)^{\beta} \left\{\left(\frac{d}{dx}\right)^{\mu} x^m\right\},$$

by (7)
$$= \left(\frac{d}{dx}\right)^{\beta} \{B_\mu x^{m-\mu}\},$$

therefore
$$\left(\frac{d}{dx}\right)^{\lambda} x^m = B^\mu \left(\frac{d}{dx}\right)^{\beta} x^{m-\mu} \;\ldots\ldots\ldots (13).$$

Since β is a positive whole number, the operation $\left(\frac{d}{dx}\right)^{\beta} x^{m-\mu}$ is one purely of common successive differentiation; hence the general differential coefficient of $\left(\frac{d}{dx}\right)^{\lambda} x^m$ can be assigned when λ is positive.

(2) Again, let λ be negative; then

$$\left(\frac{d}{dx}\right)^{-\lambda} = \left(\frac{d}{dx}\right)^{-\beta-\mu}.$$

Hence, $$\left(\frac{d}{dx}\right)^{-\lambda} x^m = \left(\frac{d}{dx}\right)^{-\beta-\mu} x^m$$

$$= \left(\frac{d}{dx}\right)^{-\beta}\left\{\left(\frac{d}{dx}\right)^{-\mu} x^m\right\}$$

by (8), $$= \left(\frac{d}{dx}\right)^{-\beta}\{B_{-\mu} x^{m+\mu}\},$$

therefore $$\left(\frac{d}{dx}\right)^{-\lambda} x^m = B_{-\mu}\left(\frac{d}{dx}\right)^{-\beta} x^{m+\mu} \text{ .... (14).}$$

Now the operation $\left(\frac{d}{dx}\right)^{-\beta} x^{m+\mu}$ is one purely of common successive integration; hence the general differential coefficient of $\left(\frac{d}{dx}\right)^{-\lambda} x^m$ can be assigned.

(3) In precisely the same way, by means of (11) and (12), can the differential coefficients of $\left(\frac{d}{dx}\right)^{\lambda} x^{-m}$ be assigned for λ, either positive or negative.

It will be convenient to classify these expressions for the general differential coefficients according as m is positive or negative. Thus,

$$\left(\frac{d}{dx}\right)^{\lambda} x^m = B_{\mu}.\left(\frac{d}{dx}\right)^{\beta} x^{m-\mu}\,;\;\; \left(\frac{d}{dx}\right)^{-\lambda} x^m = B_{-\mu}.\left(\frac{d}{dx}\right)^{-\beta} x^{m+\mu} \text{ ...(D),}$$

$$\left(\frac{d}{dx}\right)^{\lambda} x^{-m} = C_{\mu}.\left(\frac{d}{dx}\right)^{\beta} x^{-m-\mu};\;\; \left(\frac{d}{dx}\right)^{-\lambda} x^{-m} = C_{-\mu}.\left(\frac{d}{dx}\right)^{-\beta} x^{-m+\mu} \text{...(E).}$$

§ 3. To investigate expressions for

$$\left(\frac{d}{dx}\right)^{\beta} x^{m-\mu} \text{ and } \left(\frac{d}{dx}\right)^{-\beta} x^{m-\mu}$$

within the limits of Legendre's function.

(1) We begin with $\left(\frac{d}{dx}\right)^{-\beta} x^{m+\mu}$. And since β is a whole number, we have, as in (1),

$$\left(\frac{d}{dx}\right)^{-\beta} x^{m+\mu} = \frac{\Gamma(1+m+\mu)}{\Gamma(1+m+\mu+\beta)}\, x^{m+\mu+\beta}$$

$$= \frac{\Gamma(1+m+\mu)}{\Gamma(1+m+\lambda)}\, x^{m+\lambda} \;\ldots\ldots\ldots\ldots(15).$$

$$= \frac{\Gamma(1+m+\mu)}{\Gamma(1+n+p+\beta+\mu)}\, x^{m+\lambda} \;\ldots\ldots(16).$$

(2) The signs of λ, β, μ change together, since $\lambda = \beta + \mu$. If, then, we change the sign of β in (16), we have

$$\left(\frac{d}{dx}\right)^{\beta} x^{m-\mu} = \frac{\Gamma(1+m-\mu)}{\Gamma(1+n+p-\beta-\mu)}\, x^{m-\lambda} \;\ldots\ldots(17),$$

$$= \frac{\Gamma(1+m-\mu)}{\Gamma(1+m-\lambda)}\, x^{m-\lambda} \;\ldots\ldots\ldots\ldots(18).$$

On examining (17), we at once see that it is within the limits of Legendre's function, so long as $B \leqq n$.

In the particular case when $\beta = n$, we have, by (17),

$$\left(\frac{d}{dx}\right)^{n} x^{m-\mu} = \frac{\Gamma(1+m-\mu)}{\Gamma(1+p-\mu)}\, x^{p-\mu} \;\ldots\ldots\ldots\ldots(19).$$

(3) Let $\beta > n$, and put $\beta = n + \beta_1$, when β_1 will be a positive whole number; and $\left(\frac{d}{dx}\right)^{\beta} = \left(\frac{d}{dx}\right)^{n+\beta_1}$; hence

$$\left(\frac{d}{dx}\right)^{\beta} x^{m-\mu} = \left(\frac{d}{dx}\right)^{n+\beta_1} x^{m-\mu}$$

$$= \left(\frac{d}{dx}\right)^{\beta_1} \left\{\left(\frac{d}{dx}\right)^{x} x^{m-\mu}\right\}$$

by (19),
$$= \left(\frac{d}{dx}\right)^{\beta_1} \left\{\frac{\Gamma(1+m-\mu)}{\Gamma(1+p-\mu)}\, x^{p-\mu}\right\}$$

therefore
$$\left(\frac{d}{dx}\right)^{\beta} x^{m-\mu} = \frac{\Gamma(1+m-\mu)}{\Gamma(1+p-\mu)} \left(\frac{d}{dx}\right)^{\beta_1} x^{p-\mu} \;\ldots\ldots\ldots(20).$$

Since p and μ are both positive proper fractions, their difference $(p - \mu)$ will also be a proper fraction either positive or negative. First, let $(p - \mu)$ be positive, and put $q = 1 - (p - \mu)$; hence $(p - \mu) = 1 - q$, which substitute in

(20); and

$$\left(\frac{d}{dx}\right)^{\beta} x^{m-\mu} = \frac{\Gamma(1+m-\mu)}{\Gamma(1+p-\mu)} \left(\frac{d}{dx}\right)^{\beta_1} x^{1-q}.$$

Operate on both sides of this with $\left(\frac{d}{dx}\right)$, and

$$\left(\frac{d}{dx}\right)^{\beta+1} x^{m-\mu} = \frac{\Gamma(1+m-\mu)}{\Gamma(1+p-\mu)} \left(\frac{d}{dx}\right)^{\beta_1+1} x^{1-q}$$

$$= \frac{\Gamma(1+m-\mu)}{\Gamma(1+p-\mu)} \left(\frac{d}{dx}\right)^{\beta_1} \left\{\left(\frac{d}{dx}\right) x^{1-q}\right\}$$

$$= \frac{\Gamma(1+m-\mu)}{\Gamma(1+p-\mu)} \left(\frac{d}{dx}\right)^{\beta_1} \{(1-q)\, x^{-q}\}$$

$$= (1-q) \frac{\Gamma(1+m-\mu)}{\Gamma(1+p-\mu)} \left(\frac{d}{dx}\right)^{\beta_1} x^{-q}$$

by (A), $$= (1-q) \frac{\Gamma(1+m-\mu)}{\Gamma(1+p-\mu)} \left\{(-1)^{\beta_1} \frac{\Gamma(q+\beta_1)}{\Gamma(q)} x^{-q-\beta_1}\right\}$$

$$= (-1)^{\beta_1} \frac{(1-q)\,\Gamma(q+\beta_1)\,\Gamma(1+m-\mu)}{\Gamma(q)\,\Gamma(1+p-\mu)} x^{-q-\beta_1}$$

replace $(p-\mu)$, $$= (-1)^{\beta-n} \frac{(p-\mu)\,\Gamma(1-p+\mu+\beta-n)\,\Gamma(1+m-\mu)}{\Gamma(1-p+\mu)\,\Gamma(1+p-\mu)} x^{p+n-\beta-\mu-1}$$

$$\therefore \left(\frac{d}{dx}\right)^{\beta+1} x^{m-\mu} = (-1)^{\beta-n} \frac{(p-\mu)\,\Gamma(1-m+\lambda)\,\Gamma(1+m-\mu)}{\Gamma(1-p+\mu)\,\Gamma(1+p-\mu)} x^{m-\lambda-1}.$$

Operate on both sides of this with $\left(\frac{d}{dx}\right)^{-1}$, and

$$\left(\frac{d}{dx}\right)^{\beta} x^{m-\mu} = (-1)^{\beta-n} \frac{(p-\mu\,\Gamma(1-m+\lambda)\,\Gamma(1+m-\mu)}{(m-\lambda)\,\Gamma(1-p+\mu)\,\Gamma(1+p-\mu)} x^{m-\lambda} \ldots (21).$$

This expression is within the limit of Legendre's function, when $\beta > n$.

§ 4. To investigate expressions for

$$\left(\frac{d}{dx}\right)^{\beta} x^{-m-\mu} \text{ and } \left(\frac{d}{dx}\right)^{-\beta} x^{-m+\mu}$$

within the limits of Legendre's function.

(1) We have by (A),

$$\left(\frac{d}{dx}\right)^{\beta} x^{-m-\mu} = (-1)^{\beta} \frac{\Gamma(m+\beta+\mu)}{\Gamma(m+\mu)} x^{-m-\beta-\mu}$$

$$= (-1)^{\beta} \frac{\Gamma(m+\lambda)}{\Gamma(m+\mu)} x^{-m-\lambda} \quad \ldots\ldots\ldots\ldots\ldots\ldots (22),$$

by (L),

$$= (-1)^{\beta} \frac{\Gamma(p+n+\beta+\mu+1)\,(m+\mu)}{(m+\mu+\beta)\,\Gamma(1+m+\mu)} x^{-m-\lambda} \ldots (23).$$

(2) If we change the signs of λ, β, μ in (23), we have

$$\left(\frac{d}{dx}\right)^{-\beta} x^{-m+\mu} = (-1)^{-\beta} \frac{(m-\mu)\,\Gamma(1+p+n-\beta-\mu)}{(m-\beta-\mu)\,\Gamma(1+m-\mu)} x^{-m+\lambda} \ldots (24),$$

since $(-1)^{-\beta} = (-1)^{\beta}$, β being integer,

$$= (-1)^{\beta} \frac{(m-\mu)\,\Gamma(1+m-\lambda)}{(m-\lambda)\,\Gamma(1+m-\mu)} x^{-m+\lambda} \quad \ldots\ldots\ldots (25).$$

We instantly see that (24) is within the limits of Legendre's function, so long as $\beta \leqq n$. In the particular case when $\beta = n$, we have

$$\left(\frac{d}{dx}\right)^{-n} x^{-m+\mu} = (-1)^{n} \frac{(m-\mu)\,\Gamma(1+p-\mu)}{(p-\mu)\,\Gamma(1+m-\mu)} x^{\mu-p} \ldots (26).$$

(3) When $\beta > n$, put $\beta = n + \beta_1$, when β_1 will be a whole number, therefore

$$\left(\frac{d}{dx}\right)^{-\beta} x^{-m+\mu} = \left(\frac{d}{dx}\right)^{-\beta_1-n} x^{-m+\mu}$$

$$= \left(\frac{d}{dx}\right)^{-\beta_1} \left\{\left(\frac{d}{dx}\right)^{-n} x^{-m+\mu}\right\},$$

by (26),

$$= \left(\frac{d}{dx}\right)^{-\beta_1} \left\{(-1)^{n} \frac{(m-\mu)\,\Gamma(1+p-\mu)}{(p-\mu)\,\Gamma(1+m-\mu)} x^{\mu-p}\right\}$$

$$= (-1)^{n} \frac{(m-\mu)\,\Gamma(1+p-\mu)}{(p-\mu)\,\Gamma(1+m-\mu)} \left(\frac{d}{dx}\right)^{-\beta_1} x^{\mu-p},$$

by (1),

$$= (-1)^{n} \frac{(m-\mu)\,\Gamma(1+p-\mu)\,\Gamma(1-p+\mu)}{(p-\mu)\,\Gamma(1+m-\mu)\,\Gamma(1-p+\mu+\beta_1)} x^{\mu-p+\beta_1}$$

$$\therefore \left(\frac{d}{dx}\right)^{-\beta} x^{-m+\mu} = (-1)^{n} \frac{(m+\mu)\,\Gamma(1+p-\mu)\,\Gamma(1-p+\mu)}{(p-\mu)\,\Gamma(1+m-\mu)\,\Gamma(1-m+\lambda)} x^{-m+\lambda} \ldots (27).$$

And this is the expression required, when $\beta > n$.

The foregoing equations comprehend all possible cases.

§ 5. I shall now classify the expressions here obtained for the general differential coefficients, writing them at full length after expunging the factors, common both to their numerators and denominators. This will bring them immediately under the eye, and give a greater facility of reduction to those who may choose to take the trouble of reducing them.

Cases of (D).

I. The form $\left(\frac{d}{dx}\right)^{\lambda} x^{m} = B_{\mu} \left(\frac{d}{dx}\right)^{\beta} x^{m-\mu}$ presents the two following cases.

(1) When $\beta > n$; then by (18), we have

$$\left(\frac{d}{dx}\right)^{\lambda} x^{m} = (-1)^{\mu} . \frac{p . \lceil\overline{1-p+\mu}\, \lceil\overline{1+p-\mu}\, \lceil\overline{1+m}}{(p-\mu) . \lceil\overline{1-p}\, \lceil\overline{1+p}\, \lceil\overline{1+m-\lambda}} x^{m-\lambda} \ldots (a).$$

(2) When $\beta > n$; then by (21), we have

$$\left(\frac{d}{dx}\right)^{\lambda} x^{m} = (-1)^{\lambda-n} . \frac{p . \lceil\overline{1+m}\, \lceil\overline{1-m+\lambda}}{(m-\lambda)\, \lceil\overline{1-p}\, \lceil\overline{1+p}} x^{m-\lambda} \ldots . (b).$$

II. The form $\left(\frac{d}{dx}\right)^{-\lambda} x^{m} = B_{-\mu} \left(\frac{d}{dx}\right)^{-\beta} x^{m+\mu}$ presents only the following case; for by (15), we have

$$\left(\frac{d}{dx}\right)^{-\lambda} x^{m} = (-1)^{-\mu} . \frac{p . \lceil\overline{2-p-\mu}\, \lceil\overline{1+p+\mu}\, \lceil\overline{1+m}}{(p+\mu)(1-p-\mu) \lceil\overline{1-p}\, \lceil\overline{1+p}\, \lceil\overline{1+m+\lambda}} x^{m+\lambda} \ldots (c).$$

Cases of (E).

III. The form $\left(\frac{d}{dx}\right)^{\lambda} x^{-m} = C_{\mu} \left(\frac{d}{dx}\right)^{\beta} x^{-m-\mu}$ presents but one case; for by (22), or at once by (A),

$$\left(\frac{d}{dx}\right)^{\lambda} x^{-m} = (-1)^{\lambda} . \frac{\lceil\overline{m+\lambda}}{\lceil\overline{m}} x^{-m-\lambda} \ldots \ldots \ldots (d).$$

IV. The form $\left(\frac{d}{dx}\right)^{-\lambda} x^{-m} = C_{-\mu} \left(\frac{d}{dx}\right)^{-\beta} x^{-m+\mu}$ presents two cases.

(1) When $\beta > n$; then by (25), we have

$$\left(\frac{d}{dx}\right)^{-\lambda} x^{-m} = (-1)^{-\lambda} . \frac{\lceil\overline{1+m-\lambda}}{(m-\lambda)\, \lceil\overline{m}} x^{-m+\lambda} \ldots \ldots (e).$$

(2) When $\beta > n$; then, by (27), we have

$$\left(\frac{d}{dx}\right)^{-\lambda} x^{-m} = (-1)^{n-\mu}.\frac{\Gamma(1-p+\mu)\,\Gamma(1+p-\mu)}{(p-\mu)\,\Gamma m\,\Gamma(1-m+\lambda)}\,x^{-m+\lambda}\ldots(f).$$

Notation of the foregoing Classified.

Here λ and m represent any positive numbers whatever;
β and n any positive *whole* numbers;
μ and p any positive proper fractions;
with the equations of conditions $\lambda = \beta + \mu$ and $m = n + p$.

The expressions here given for the general differential coefficients appear to have the simplest forms of which they are capable, so as to preserve the rigour of demonstration, and at the same time to render them *immediately* interpretable. They may be all reduced to *one universal* form by a *mere extension* of the Γ notation beyond the limit at which the *real* function ceases to exist. This reduction will be given in a subsequent paper. But the subject of the present paper is complete without such reduction.

[*To be completed in the next Number.*]

Longside, Mintlaw, October 30, 1847.

MATHEMATICAL NOTES.

*I.—Demonstration of Pascal's Hexagramme.**

By Thomas Weddle.

Let $ABCDEF$ be a hexagon inscribed in a conic section, and let $u = 0\ldots(1)$, $v = 0\ldots(2)$, $w = 0\ldots(3)$,

be the equations to the alternate sides AB, CD and EF. Supposing u, v, and w to have been multiplied by arbitrary constants, we may denote the equation to the conic section by

$$u^2 + v^2 + w^2 - (\lambda + \lambda^{-1})\,vw - (\mu + \mu^{-1})\,wu - (\nu + \nu^{-1})\,uv = 0\ldots(4).$$

---

* In a letter to the Editor, Mr. Weddle gives the following references for proofs of Pascal's Hexagramme.

Two by Mr. Cayley; one (from Chasles) *Journal*, Old Series, vol. III. p. 211; the other, vol. IV. p. 18.

In the *Ladies and Gentlemen's Diary* for 1843, proofs are given by Messrs. Gill, Fenwick, and Dunelmensis (Mr. William Finlay, Durham).

The Mathematician, vol. I. p. 132, by Mr. Fenwick; vol. II. p. 15, by Mr. Weddle.

Salmon's *Conic Sections*, p. 212.

In Davies's *Hutton*, vol. II. pp. 213-313, demonstrations are given by Mr. Rutherford and by Mr. Fenwick; which have also been contributed to the *Philosophical Magazine*.

All these differ materially from the demonstration in the text.]

To determine the points A and B, put $u = 0$, therefore

$$v^2 + w^2 - (\lambda + \lambda^{-1})\, vw = 0, \qquad \text{hence } v = \lambda w \text{ or } w = \lambda v;$$

and, in like manner, by successively putting $v = 0$ and $w = 0$ in (4), we shall get the equations that determine the positions of the other angular points of the hexagon. The points A, B, C, D, E and F are therefore denoted as follows:

$$
\begin{array}{lll}
A \text{ by } u = 0, & v = \lambda w \ldots\ldots\ldots\ldots & (5), \\
B \;\ldots\; u = 0, & w = \lambda v \ldots\ldots\ldots\ldots & (6), \\
C \;\ldots\; v = 0, & w = \mu u \ldots\ldots\ldots\ldots & (7), \\
D \;\ldots\; v = 0, & u = \mu w \ldots\ldots\ldots\ldots & (8), \\
E \;\ldots\; w = 0, & u = \nu v \ldots\ldots\ldots\ldots & (9), \\
F \;\ldots\; w = 0, & v = \nu u \ldots\ldots\ldots\ldots & (10).
\end{array}
$$

Hence (8, 9), (10, 5), and (6, 7), the equations to the lines DE, FA and BC are, by mere inspection, found to be

$$
\begin{array}{ll}
u = \mu w + \nu v \ldots\ldots\ldots\ldots\ldots & (11), \\
v = \lambda w + \nu u \ldots\ldots\ldots\ldots\ldots & (12), \\
w = \lambda v + \mu u \ldots\ldots\ldots\ldots\ldots & (13).
\end{array}
$$

Now each of the three pairs of equations (1, 11), (2, 12) and (3, 13) evidently satisfies the equation

$$\frac{u}{\lambda} + \frac{v}{\mu} + \frac{w}{\nu} = 0 \ldots\ldots\ldots\ldots\ldots (14),$$

and hence the opposite sides of the hexagon $ABCDEF$ intersect in three points in a straight line.

Wimbledon, April 3, 1848.

II.—On an Integral Transformation.

By Arthur Cayley.

The following transformation, given for elliptic functions by Guderman (*Crelle*, tom. XXIII. p. 330) is useful for some other integrals.

If $$y = \frac{dbc - dba - dca + abc - (bc - ad)\, z}{(bc - ad) + (d - b - c + a)\, z},$$

then, putting $K = (bc - ad) + (d - b - c + a)\, z$,

we have, supposing $a < b < c < d$, so that $(b - a)$, $(c - a)$, $(d - b)$, $(d - c)$ are positive,

$$
\begin{aligned}
K(y - a) &= (b - a)(c - a)(d - z), \\
K(y - b) &= (b - a)(d - b)(c - z), \\
K(y - c) &= (c - a)(d - c)(b - z), \\
K(y - d) &= (d - b)(d - c)(a - z), \\
K^2 dy &= -(b - a)(c - a)(d - b)(d - c)\, dz.
\end{aligned}
$$

In particular, if $\alpha+\beta+\gamma+\delta=-2$,

$$(y-a)^{\alpha}(y-b)^{\beta}(y-c)^{\gamma}(y-d)^{\delta}\,dy=-M(z-a)^{\delta}(z-b)^{\gamma}(z-c)^{\beta}(z-d)^{\alpha}\,dz.$$

where $M=(b-a)^{\alpha+\beta+1}(c-a)^{\alpha+\gamma+1}(d-b)^{\beta+\delta+1}(d-c)^{\gamma+\delta+1}$.

Thus, if $\alpha=\beta=\gamma=\delta=-\frac{1}{2}$,

$$\frac{dy}{\{-(y-a)(y-b)(y-c)(y-d)\}^{\frac{1}{2}}}=\frac{-dz}{\{-(z-a)(z-b)(z-c)(z-d)\}^{\frac{1}{2}}}.$$

In any case when $y=a$, $y=b$, the corresponding values of z are $z=d$, $z=c$; the last formula becomes by this means

$$\int_a^b\frac{dy}{\{-(y-a)(y-b)(y-c)(y-d)\}^{\frac{1}{2}}}=\int_c^d\frac{dy}{\{-(z-a)(z-b)(z-c)(z-d)\}^{\frac{1}{2}}}.$$

III.—On certain Curves traced on the Surface of an Ellipsoid [Poinsot's Poloids.]

To the Editor of the Cambridge and Dublin Mathematical Journal.

THE following theorems, relating to certain remarkable curves which may be traced on the surface of an Ellipsoid, and which have not, as far as I am aware, been before published, may perhaps interest some of your readers.*

1. The locus of points on an ellipsoid for which the *measure of curvature*† is constant (suppose equal to k) is a curve of double curvature, such that the tangent planes to the surface along it are at a constant distance from the centre.

2. The developable surface generated by the ultimate intersections of all these tangent planes intersects either principal plane passing through the mean axis in a conic, such that the cylinders standing on it and having their generatrices parallel to the asymptotes of the focal hyperbola are of revolution.

3. The sum or difference of the angles which each 'arète' of this developable makes with the asymptotes of the focal hyperbola is constant.

4. Hence the central sections parallel to the tangent planes of this developable will envelope a cone of the second degree, whose focal lines are the asymptotes of the focal hyperbola.

5. By varying k we obtain a system of curves on the ellipsoid along each of which the measure of curvature is

---

* The first of the theorems enunciated in the text is probably familiar to many of the readers of the *Journal;* but the three which follow it appear to be entirely new.

† By the "measure of curvature" I understand the product of the reciprocals of the principal radii of curvature.

constant: the projections of these curves on a principal plane are similar concentric conics.

6. If the same curves be projected on a cyclic plane of the ellipsoid by lines parallel to the greatest or least axis, the projections will of course be a series of similar conics; and their common eccentricity will be equal to that of the principal section of the ellipsoid perpendicular to that axis.*

GEORGE J. ALLMAN.

Trinity College, Dublin, May 3, 1848.

[If a rigid body be in motion round its centre of gravity, under the action of no forces, it will, as Poinsot demonstrates, move in such a manner that a tangent plane to the *momental ellipsoid* (see *Journal*, 2nd Series, vol. I. p. 202) at the point where its surface is cut by the instantaneous axis will be fixed in space, and parallel to that plane through the centre of gravity known as "the invariable plane."

1. Hence the curve called by Poinsot "the poloid," that is the curve traced on the surface of the momental ellipsoid by the instantaneous axis during the motion of the body, possesses, according to Mr. Allman's first theorem, the property that along it the "measure of curvature" of the surface is constant.

From Mr. Allman's other theorems we make the following inferences.

2. Any fixed plane parallel to "the invariable plane" cuts either principal plane of the momental ellipsoid containing the mean axis, in a line which, during the motion, touches an ellipse in this plane (fixed with reference to the body) such that the cylinders standing upon it, with their axes parallel to the asymptotes of the focal hyperbola, are of revolution.

3. A line drawn from any point of this ellipse to the corresponding point in the poloid, makes angles with the asymptotes of the focal hyperbola, of which the sum or difference is constant.

Or; a line drawn from the point of contact to the point of intersection of the fixed tangent plane with a perpendicular to it from the centre at any instant, makes angles, with the asymptotes to the focal hyperbola, of which the sum or difference is constant.

4. The invariable plane touches a cone, fixed with reference to the body, of which the focal lines are the asymptotes of the focal hyperbola of the momental ellipsoid.

If we draw a conical surface of any kind from the greatest to the least axis and take initial positions of the instantaneous axis along it, we get different states of motion which give the entire series of the poloids; a system of curves which possess the properties enunciated by Mr. Allman in his fifth and sixth theorems.

Glasgow College, October 23, 1848.]

[* Whitley's *Translation of Poinsot's Theory of Rotatory Motion*, p. 48.]

END OF VOLUME III.

Printed by Metcalfe and Palmer, Trinity Street, Cambridge.

SCIENTIFIC JOURNALS.

Journal de Mathématiques pures et appliquées, ou Recueil Mensuel de Mémoires sur les diverses parties des Mathématiques. Publié par Joseph Liouville, Membre de l'Académie des Sciences et du Bureau des Longitudes.

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Journal für die reine und angewandte Mathematik (In zwanglosen Heften). Herausgegeben von A. L. CRELLE, Berlin. Mit thätiger Beförderung hoher Königlich-Preussischer Behörden.

BAND XXXIV. (1847.) Heft I.—Uber die Substitutionen von der ersten Ordnung und die Umformung der elliptischen Integrale in die Normalform. Von dem Herrn Professor Dr. *F. Richelot* zu Königsberg in Pr.—Recherches sur l'élimation, et sur la théorie des courbes. Par Mr. *A. Cayley* de Cambridge.—In solutionem Aequationem Algebraicarum Disquisitio. Auct. *C. J. Malmsten*, prof. math. Upsaliens.—Die Lagrangesche Formel und die Reihensummirung durch dieselbe. Von Herrn *J. Dienger*, Lehrer der Mathematik und Physik an der höhern Bürgerschule zu Sinsheim bei Heidelberg.—Fac-simile einer Handschrift von *Ferrari*, welches der Herausgeber, durch Vermittlung des Herrn Prof. *C. G. J. Jacobi*, der Güte des Herrn Grafen *Gräberg von Hemsö* zu Florenz verdankt. Heft II. Addizione alla Memoria intitolata: Nuove applicazioni del Calcolo Integrale relative a la quadratura delle superficie curve e cubatura de solidi, inscritta nel tom. 31 di questo giornale pag^{a} 12. Dal. Sig. *Barnaba Tortolini*, prof. di matematiche trasc. a l'Università di Roma.—Uber die Lichtzerstreuung in der Atmosphäre. Von Herrn Candidaten *R. Clausius* zu Berlin.—Note sur les hyperdéterminants. Par Mr. *A. Cayley* de Cambridge.—Untersuchungen über die Wahrscheinlichkeitsrechnung. Von Herrn Dr. *Ottinger*, Prof. ord. an der Universität zu Freiburg im Br. (Fortsetzung des Aufsatzes No. 16 im dritten, No. 21 im vierten Heft 26ten, No. 17 im dritten und No. 22 im vierten Heft 30ten Bandes.)—Fac-simile einer Handschrift von *Ferroni*, welches der Herausgeber durch Vermittlung des Herrn Professor *C. G. J. Jacobi*, der Güte des Herrn Grafen *Gräberg von Hemsö* zu Florenz verdankt. Heft III. Algebraische Auflösung derjenigen Gleichungen 9ten Grades, deren Wurzeln die Eigenschaft haben, dafs eine gegebene rationale und symmetrische Function $\theta(x_\lambda, x_\mu)$ je zweier Wurzeln x_λ, x_μ eine dritte Wurzel x_k giebt, so dafs gleichzeitig:

$$x_\kappa = \theta(x_\lambda, x_\mu),\ x_\lambda = \theta(x_\mu, x_\kappa),\ x_\mu = \theta(x_\kappa, x_\lambda).$$

Von Herrn Dr. *Otto Hesse*, Prof. extraord. an der Universität zu Königsberg.—Die allgemeinen unendlichen Reihen in der Analysis und ihre Darstellung in geschlossenen Ausdrücken. Von Herrn *J. Dienger*, Lehrer der Mathematik und Physik an der höhern Bürgerschule zu Sinsheim bei Heidelberg.—De criteriis quibus cognoscatur an aequatio quinti gradus irreductibilis algebraice resolvi possit. Auctore *Eduardo Luther*, phil. doctore, Regiomonti.—Einige Aufgaben aus der Combinationslehre. Von dem Herrn Lehramts-Candidaten *Weiss* zu München.—Sur quelques théorèmes de la géométrie de position. Suite du mémoire, tome XXXI. p. 213. Par M. *A. Cayley* de Cambridge.—Ein eigenthümlicher analytischer Fall bei der Theorie der Kurbel. Vom Herausgeber.—Zwei geometrische Aufgaben; nebst den Auflösungen.—Fac-

simile einer Handschrift von *Fontana*, welches der Herausgeber, durch Vermittlung des Herrn Professor *C. G. J. Jacobi*, der Güte des Herrn Grafen *Gräberg von Hemsö* zu Florenz verdankt. Heft IV. Untersuchungen über die Reihe

$$1+\frac{(1-q^{\alpha})(1-q^{\beta})}{(1-q)(1-q^{\gamma})}.x+\frac{(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})}{(1-q)(1-q^{2})(1-q^{\gamma})(1-q^{\gamma+1})}.x^{2}+\ldots.$$

Von Herrn Dr. *E. Heine*, Privatdocenten an der Universität zu Bonn.—Über Curven dritter Ordnung und analytische Beweisführung. Von dem Herrn Prof. Dr. *Plücker* zu Bonn.—Note sur le théorème de *Pascal*. Par le même. —Die analytische Geometrie der Curven auf den Flächen zweiter Ordnung und Classe. Von demselben.—Über eine neue mechanische Erzeugung der Flächen zweiter Ordnung und Classe. Von demselben.—Bemerkung zu der Abhandlung: „Die analytische Geometrie der Curven auf den Flächen zweiter Ordnung. Von demselben.—Fac-simile einer Handschrift von *Paoli*, welches der Herausgeber, durch Vermittlung des Herrn Prof. *C. G. J. Jacobi*, der Güte des Herrn Grafen *Gräberg von Hemsö* zu Florenz verdankt.

Band XXXV. (1847.) Heft I.—Beitrag zur Theorie der Function

$$\Gamma(x)=\int_0^{\infty} e^{-v} v^{x-1} dv.$$

Von Herrn Dr. *E. E. Kummer*, Professor in Breslau.—Über Systeme von Curven, welche einander überall rechtwinklig durchschneiden. Von demselben.—Untersuchungen über die analytischen Facultäten. Von dem Herrn Prof. *Ottinger* zu Freyburg i. Br. (Fortsetzung von No. 1, 7, 11, und 17. Band XXXIII.)—Sur la formule

$$hu_x'=\Delta u_x-\frac{h}{2}.\Delta u_x'+\frac{B_1h^2}{1.2}.\Delta u_x''-\frac{B_2h^4}{1..4}.\Delta u_x^{\text{IV}}+\text{etc.}$$

Par *C. J. Malmstén*, Profess. des Math. à Upsala.—Bemerkungen zu einer gewissen Methode, die Gleichung eines durch vier Puncte gehenden Kegelschnitts auszudrücken. Von dem Herrn Dr. *Arndt* in Stralsund.—Fac-simile einer Handschrift von *Viviani*, welches der Herausgeber, durch Vermittlung des Herrn Prof. *C. G. J. Jacobi*, der Güte des Herrn Grafen *Gräberg von Hemsö* zu Florenz verdankt. Heft II. Über das Ohmsche physicalische Gesetz. Vom Herrn Professor Dr. *Plücker* zu Bonn.—Sur la Réflexion de la Lumière, dans le cas des surfaces du second degré, analogue à celle qui aux foyers des sections coniques a donné le nom. Par le même.—*Leonardi Euleri* Commentatio de Matheseos sublimioris utilitate ex autographo edidit *G. Friedlaenderus*. Berolini MDCCCXLVII.—Neue Theoreme der höheren Arithmetik. Von Herrn Dr. phil. *G. Eisenstein*, Privatdocent zu Berlin.—Beiträge zur Theorie der elliptischen Functionen. Von demselben. (IV) Über einen allgemeinen Satz, welcher das Additionstheorem für elliptische Functionen als speciellen Fall enthält. (V) Über die Differentialgleichungen, welchen der Zähler und der Nenner bei den elliptischen Transformationsformeln genügen. (VI) Genaue Untersuchung der unendlichen Doppelproducte, aus welchen die elliptischen Functionen als Quotienten zusammengesetzt sind.—Fac-simile einer Handschrift von *J. F. Pfaff*, aus den Acten des Hohen Königlich-Preussischen Ministeriums der Geistlichen-, Unterrichts- und Medicinal-Angelegenheiten, mit Hochdesselben Erlaubniss hier mitgetheilt. Heft III. Beiträge zur Theorie der elliptischen Functionen. Von Herrn Dr. phil. *G. Eisenstein*, Privatdocent zu Berlin. (VI) Genaue Untersuchung der unendlichen Doppelproducte, aus welchen die elliptischen Functionen als Quotienten zusammengesetzt sind, und der mit ihnen zusammenhangenden Doppelreihen. (Fortsetzung der letzten Abhandlung im vorigen Heft.)—Aufgaben und Lehrsätze. Von demselben.—Fac-simile einer Handschrift von *Tralles*, aus den Acten des Hohen Königlich-Preussischen Ministeriums der Geistlichen-, Unterrichts- und Medicinal-Angelegenheiten, mit Hochdesselben Erlaubniss hier mitgetheilt. Heft IV. Theoriae transcendentium Abelianarum primi ordinis adumbratio levis. Auctore Dr. *A Göpel*.—Notiz über *A. Göpel*.—Zur Theorie der complexen Zahlen. Von dem Herrn Prof. Dr. *Kummer*. (Auszug aus den Berichten der Königl. Akad. der Wiss. zu Berlin vom März 1845.)—Über die Zerlegung der aus Wurzeln der Einheit gebildeten complexen Zahlen in ihre Primfactoren. Von demselben.—Note sur la représentation d'un nombre par la somme de cinq carrés. Par Mr. *G. Eisenstein*.—Fac-simile einer Handschrift von *Bessel*. Ein Brief desselben an den Herausgeber dieses Journals.

GLASGOW COLLEGE, *March* 20, 1848.

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St. Peter's College, *May* 27, 1848.

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Cambridge, May 27, 1848.

The Publishers of the *Cambridge and Dublin Mathematical Journal* regret to have to announce that the sale is not sufficient to meet the expenses, when the Numbers are supplied through the booksellers and the usual trade allowance given. They therefore propose, after the completion of the present volume, to publish the *Journal* by *Annual Subscriptions, payable in advance;* and as the recent Post-Office regulation allows of books under a pound weight going by Post for Sixpence, they propose to publish yearly Three Numbers of Six Sheets each, instead of publishing, as hitherto, the same amount of matter in Four Numbers. They will undertake to forward the Numbers *Free by Post*, as they appear, for 16*s.* 6*d.* annually. To gentlemen residing in Cambridge, or to others to whom the Numbers can be delivered without the expense of postage, the charge will be 15*s.* a-year.

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